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One axiom characterization for (L, M)-fuzzy rough approximation operators

Ayat A. Temraz *

Department of Mathematics, Faculty of Science, South Valley University, Qena, 83523, Egypt

**Email: ayat.tmraz@sci.svu.edu.eg*

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Abstract: A thorough comprehension of the single axiomatic characterization governing fuzzy rough approximation operators is crucial for delving deeper into the foundational principles of rough set theory. By analyzing these operators through an axiomatic lens, researchers can gain valuable insights into the structural and theoretical underpinnings of rough sets, enabling more rigorous exploration of their conceptual framework. This paper focuses on developing a single axiom to characterize each kind of M-level L-rough approximation operators or (L, M)-fuzzy rough approximation operators (LM-Rapprox operators for short) produced by non-increasing, unary, reflexive, serial, and transitive LM-fuzzy G neighborhood system (LM-fgns for short), as well as their compositions. Finally, we discuss the relationship between the LM-Rapprox operators and the LM-quasi fuzzy topologies. Specifically, it demonstrates that the lower and upper LM-Rapprox operators derived from LM-fgns correspond to a pair of LM-quasi-fuzzy interior and LM-fuzzy closure operators, respectively.

Keywords: Semi quantale, LM-fgns, single axiom, lower and upper LM-Rapprox operators, LM-quasi fuzzy topology.

2020 AMS Classification: 54A10, 54A05, 54A40.

1. Introduction

The rough set theory, initially presented by Pawlak **[1],** is a modern method for information processing that has been widely applied in numerous practical areas such as data analysis and data mining **[2, 3]**. Also, Alsulami et al. [4] proposed a novel form of roughness that extends several earlier definitions, both in the fuzzy framework and in the classical setting. In general, the constructive approach and the axiomatic approach are the two basic techniques for developing rough set theory by studying the upper and lower approximation operators, which represent their fundamental concepts. In the constructive approach, the lower and upper approximation operators are constructed using primitive concepts on a universe of discourse, such as binary relations [5, 6], coverings [7, 8, 9], and generalized neighborhood systems [10, 11]. In contrast, in the axiomatic approach, the abstract upper and lower approximation operators are treated as primary notions, and a collection of axioms is used to characterize the approximation operators that are generated in the constructive approach. For axiomatic characterizations of approximation operators, see the literature [12, 13, 14].

Later, researchers primarily used constructive and axiomatic techniques to describe fuzzy approximation operators based on fuzzy coverings, fuzzy binary relations, and fuzzy generalized neighborhood systems in the context of fuzzy rough set theory [15, 16, 17, 18, 19].

A fascinating mathematical question is whether approximations to rough sets can be described by a single axiom. The characterization of rough approximation operators by means of single axioms is of crucial importance for the study of the theory of crisp and fuzzy rough sets. This idea has inspired many researchers to explore single axioms for classical and fuzzy rough approximation operators, as seen in the works of Bao et al. [20], Liu [21], Wang [22], Pang et al. [23] and Wu et al. [24, 25].

In 2021, Zhao and Shi [26] used single axioms to characterize various types of L-fuzzy approximation operators corresponding to reflexive, serial, weakly transitive, and weakly unary L-fgns. In addition, Jin et al. [27] provided single axiomatic characterizations for another 11 types of LFVPRSs. Further, Chen et al [28] studied single axiomatic characterizations of L-valued rough sets.

In 2022, El-Saady et al. [29] introduced a pair of many level lower and upper rough approximation operators derived from LM-fgns. They also demonstrated that these approximation operators encompass L-fgns-based approximation operators (Zhao et al. [14]; Zhao et al. [30]; Zhao et al. [26]) and M-level L-fuzzy relation-based approximation operators (Sostak et al. [31]) as special cases.

Motivations, Innovative-ness, and Contributions

- Previous investigations have shown that the study of L-fuzzy generalized neighborhood system (L-fgns) based approximation operators remains incomplete [29]. This gap highlights the need for deeper theoretical exploration, particularly regarding the axiomatic foundations of LM-Rapprox operators derived from LM-fgns.
- One significant research challenge concerns the single-axiom characterization of approximation operators, a theme that has been widely addressed across different

generalized rough set models applied to both crisp and fuzzy rough set theory. However, El-Saady et al. **[29]** did not provide a single-axiom characterization for his proposed lower and upper approximation operators. Thus, the first objective of this work is to present LM-Rapprox operators within the LM-fgns framework using a single-axiom approach.

- In addition, previous studies have examined the relationship between fuzzy approximation operators and fuzzy topological structures within fuzzy rough set theory **[32, 33, 34, 26]**. Building on this line of research, the second aim of this article is to explore the interplay between LM-quasi-fuzzy topologies and LM-Rapprox operators.
- Beyond theoretical significance, lower and upper approximation operators have been successfully applied in several real-world scenarios, including multi-criteria decision-making techniques (SAW, TOPSIS, PROMETHEE) for evaluating auction systems in procurement processes [35], decision support systems [36, 37], and the assessment of online health information quality [38].

The following is a breakdown of the paper's structure. In Section 2, we review several fundamental concepts and notes. Section 3 shows that the lower and upper LM-Rapprox operators generated by reflexive, non-increasing, serial, transitive, and unary LM-fgns, can be described by single axioms, also we discuss the relationship between LM-quasi-fuzzy topologies are discussed and LM-Rapprox operators based on LM-fgns. The final section summarizes our findings, offering thoughtful conclusions based on the study conducted.

2. Preliminaries

A structure $\mathrm{L}=(\mathrm{L},\leq)$, which is a complete lattice and is endowed with a binary operation $\otimes:\mathrm{L}\times\mathrm{L}\to\mathrm{L}$ without imposing any further restrictions, is referred to as a semi-quantale (abbreviated as s-quantale) $\mathrm{L}=(\mathrm{L},\leq,\otimes)$ [39]. As an agreement, we indicate the meet, join, bottom and top in $\mathrm{L}=(\mathrm{L},\leq)$ by $\wedge,\vee,\perp_{\mathrm{L}}$ *and* $\top_{\mathrm{L}}$, respectively.

Definition 1. A s-quantale $\mathrm{L}=(\mathrm{L},\leq,\otimes)$ is called

• A unital [39] if L has an identity element e for $\otimes$. If the identity is the top element of L, then a unital s-quantale is said to be a strictly two-sided s-quantale (st-s-quantale for short).

• A commutative [39] if $\otimes$ is commutative.

• A quantale [40] if $\otimes$ is associative and satisfies:

$$\eta\otimes\left(\vee_{j\in J}\zeta_j\right)=\bigvee_{j\in J}\left(\eta\otimes\zeta_j\right) \text{ and}$$

$$\left(\bigvee_{j\in J}\zeta_j\right)\otimes\eta=\bigvee_{j\in J}\left(\zeta_j\otimes\eta\right)\ \forall\,\eta\in\mathrm{L},\{\zeta_j:j\in J\}\subseteq\mathrm{L}.$$

In this article, we considered that $\mathrm{L}=(\mathrm{L},\leq,\otimes)$ is a st-s-quantale and $(\mathrm{M},\leq,\odot)$ be a s-quantale.

Consider L is a s-quantale and X be a nonempty set. We refer to a mapping $\mathrm{U}:\mathrm{X}\to\mathrm{L}$ as an L-fuzzy subset (or L-subset) of X. Also, we indicate the family of all L-fuzzy subsets on X by L^{X}. Extensions of the algebraic structures of lattice-theoretic are possible, pointwise, from the s-quantale L to L^{X}, for all $x\in\mathrm{X}$:

$$(\mathrm{U}\otimes\mathrm{V})(x)=\mathrm{U}(x)\otimes\mathrm{V}(x).$$

$$\left(\vee_{j\in J}\mathrm{U}_j\right)(x)=\vee_{j\in J}\mathrm{U}_j\,(x).$$

The powerset L^{X} is a s-quantale with respect to $\otimes$ and arbitrary sups. The elements $\underline{\perp}$ and $\underline{\top}$ represent the smallest and largest elements in L^{X}, respectively.

It is known that every commutative quantale L is induced by its $\otimes$ distributes over arbitrary joins. The function $\eta\otimes(-):\mathrm{L}\to\mathrm{L}$ has the right adjoint $\eta\to(-):\mathrm{L}\to\mathrm{L}$ introduced by $\eta\to\zeta=\vee\{\xi:\eta\otimes\xi\leq\zeta\}$. The residual $\to:\mathrm{L}\times\mathrm{L}\to\mathrm{L}$ on L holding the axiom below

$$\eta\otimes\zeta\leq\xi\Leftrightarrow\eta\leq\zeta\to\xi.$$

The residual $\to:\mathrm{L}\times\mathrm{L}\to\mathrm{L}$ on L can be expanded pointwisely to the powerset L^{X} as $\to:\mathrm{L}^{\mathrm{X}}\times\mathrm{L}^{\mathrm{X}}\to\mathrm{L}^{\mathrm{X}}$, where

$$(\mathrm{U}\to\mathrm{V})(x)=\mathrm{U}(x)\to\mathrm{V}(x).$$

Assume that L is a commutative quantale. It is considered to follow the double negation principal [20] if

$$(\eta\to\perp)\to\perp=\eta,\text{ for all }\eta\in\mathrm{L}$$

For naivety, we use $\neg\eta$ to denote $\eta\to\perp$. Also, for any $\eta,\zeta\in\mathrm{L}$, we say $\eta\oplus\zeta=\neg(\neg\eta\otimes\neg\zeta)$.

Definition 2. Presume U, V are L-fuzzy sets in X.

(1) In **[41]**, the subsethood degree $S:\mathrm{L}^{\mathrm{X}}\times\mathrm{L}^{\mathrm{X}}\to\mathrm{L}$, of U, V say $S(\mathrm{U},\mathrm{V})$ is introduced by

$$S(\mathrm{U},\mathrm{V})=\bigwedge_{x\in\mathrm{X}}\left(\mathrm{U}(x)\to\mathrm{V}(x)\right)$$

(2) In [42], the degree of intersection of U, V denoted by

$$T(\mathrm{U},\mathrm{V})=\vee_{x\in\mathrm{X}}\left(\mathrm{U}(x)\otimes\mathrm{V}(x)\right).$$

Lemma 1. [**41, 43,44**] If $\mathrm{L}=(\mathrm{L},\leq,\otimes)$ is a st-s-quantale, then $\mathrm{U}\leq\mathrm{V}$ implies $S(\mathrm{U},\mathrm{V})=\top,\ \forall\ \mathrm{U},\mathrm{V}\in\mathrm{L}^{\mathrm{X}}$. The following definitions discuss a pair of many level upper and lower rough approximation operators derived from LM-fgns. These approximation operators also encompass L-fuzzy generalized neighborhood system-based approximation operators [**26, 30, 45**] and M-level L-fuzzy relation-based approximation operators [**31**] as special cases.

Definition 3. [29] Presume X is the universe of discourse. An LM-fgns operator on X is a mapping $\mathrm{N}:\mathrm{X}\to\mathrm{L}^{\mathrm{L}^{\mathrm{X}}\times\mathrm{M}}$, where for each $x\in\mathrm{X}$, $\mathrm{N}(x)=\mathrm{N}_x:\mathrm{L}^{\mathrm{X}}\times\mathrm{M}\to\mathrm{L}$ is non empty, i.e., $\bigvee_{\mathrm{U}\in\mathrm{L}^{\mathrm{X}}}\mathrm{N}_x(\mathrm{U},\Theta)=\top_{\mathrm{L}}$ for all $\Theta\in\mathrm{M}$. Additionally, N_x is referred to as an LM-fgns of x, and $\mathrm{N}_x(\mathrm{U},\Theta)$ represents the degree of that $\mathrm{U}\in\mathrm{L}^{\mathrm{X}}$ is considered a neighborhood of x.

Example 1. [29] Let $\mathrm{X}=\{x\}$ and $\mathrm{L}=\mathrm{M}=[0,1]$. Define an LM-fgns operator $N:\mathrm{X}\to\mathrm{L}^{\mathrm{L}^{\mathrm{X}}\times\mathrm{M}}$ by

$$\mathrm{N}_x(\mathrm{U},\Theta)=\begin{cases}1 & \textit{for}\quad \mathrm{U}=1_{\mathrm{X}};\\ \dfrac{2\Theta}{1+2\Theta} & \textit{for}\quad \mathrm{U}=x_{\frac{1}{2}};\\ \dfrac{3\Theta}{2+3\Theta} & \textit{for}\quad \mathrm{U}=x_{\frac{1}{3}};\\ 0 & \textit{otherwise}.\end{cases}$$

It's simple to observe that $\mathrm{N}:\mathrm{X}\to\mathrm{L}^{\mathrm{L}^{\mathrm{X}}\times\mathrm{M}}$ is an LM-fgns operator.

Remark 1. For all $\Theta\in\mathrm{M}$, we observe that every LM-fgns operator $\mathrm{N}:\mathrm{X}\to\mathrm{L}^{\mathrm{L}^{\mathrm{X}}\times\mathrm{M}}$ induces an L-fgns operator $\mathrm{N}_\Theta:\mathrm{X}\to\mathrm{L}^{\mathrm{L}^{\mathrm{X}}}$ [**30, 45**] given by $\mathrm{N}_\Theta=(\mathrm{N}_x^\Theta)_{x\in\mathrm{X}}$ where $\mathrm{N}_x^\Theta:\mathrm{L}^{\mathrm{X}}\to\mathrm{L}$, $\mathrm{N}_x^\Theta(\mathrm{U})=\mathrm{N}_x(\mathrm{U},\Theta),\ \forall\,\mathrm{U}\in\mathrm{L}^{\mathrm{X}}$.

Example 2. [29] For $\Theta\in\left\{\frac{2}{3},\frac{1}{2}\right\}$ and with the LM-fgns operator $\mathrm{N}:\mathrm{X}\to\mathrm{L}^{\mathrm{L}^{\mathrm{X}}\times\mathrm{M}}$ given in **Example 1**, then we have the next three L-fgns operators:

$$\mathrm{N}^{\left(\frac{2}{3}\right)}(x)(\mathrm{U}) = \mathrm{N}_x\left(\mathrm{U}, \frac{2}{3}\right) = \begin{cases} 1 & for \quad \mathrm{U} = 1_{\mathrm{X}}; \\ \frac{4}{7} & for \quad \mathrm{U} = x_{\frac{1}{2}}; \\ \frac{1}{2} & for \quad \mathrm{U} = x_{\frac{1}{3}}; \\ 0 & otherwise. \end{cases}$$

$$\mathrm{N}^{\left(\frac{1}{2}\right)}(x)(\mathrm{U}) = \mathrm{N}_x\left(\mathrm{U}, \frac{1}{2}\right) = \begin{cases} 1 & for \quad \mathrm{U} = 1_{\mathrm{X}}; \\ \frac{1}{2} & for \quad \mathrm{U} = x_{\frac{1}{2}}; \\ \frac{3}{7} & for \quad \mathrm{U} = x_{\frac{1}{3}}; \\ 0 & otherwise. \end{cases}$$

Definition 4. [29] Let $\mathrm{N}: \mathrm{X} \to \mathrm{L}^{\mathrm{L}^{\mathrm{X}} \times \mathrm{M}}$ be an LM-fgns operator. Then the lower LM-Rapprox operator $\underline{\mathrm{N}}: \mathrm{L}^{\mathrm{X}} \times \mathrm{M} \to \mathrm{L}^{\mathrm{X}}$ and the upper LM-Rapprox operator $\overline{\mathrm{N}}: \mathrm{L}^{\mathrm{X}} \times \mathrm{M} \to \mathrm{L}^{\mathrm{X}}$ are defined as follows: for each $\mathrm{U} \in \mathrm{L}^{\mathrm{X}}$, $\forall\, \Theta \in \mathrm{M}$, and $\forall\, x \in \mathrm{X}$,

$$\underline{\mathrm{N}}(\mathrm{U}, \Theta)(x) = \bigvee_{K \in \mathrm{L}^{\mathrm{X}}} \left(\mathrm{N}_x(K, \Theta) \otimes S(K, \mathrm{U})\right),$$

$$\overline{\mathrm{N}}(\mathrm{U}, \Theta)(x) = \bigwedge_{K \in \mathrm{L}^{\mathrm{X}}} \left(\mathrm{N}_x(K, \Theta) \to T(K, \mathrm{U})\right).$$

Example 3. [29] Let $\mathrm{X} = \{x\}$ and $\mathrm{L} = \mathrm{M} = [0,1]$ with the adjoint pair $(*, \to)$ on $[0,1]$ define as follows: $\forall\, \eta, \zeta \in \mathrm{L}$,

$$\eta * \zeta = \max\{0, \eta + \zeta - 1\}, \quad \eta \to \zeta = \min\{1, 1 - \eta + \zeta\}.$$

For an LM-fgns operator $\mathrm{N}: \mathrm{X} \to L^{\mathrm{L}^{\mathrm{X}} \times \mathrm{M}}$, given in **Example 1**, the lower LM-Rapprox operator $\underline{\mathrm{N}}$ given by: For $\mathrm{U} = x_{\frac{1}{3}}$

$$\begin{aligned}
\underline{\mathrm{N}}\left(x_{\frac{1}{3}}, \Theta\right)(x) &= \bigvee_{K \in \mathrm{L}^{\mathrm{X}}} \left(\mathrm{N}_x(K, \Theta) * S\left(K, x_{\frac{1}{3}}\right)\right) \\
&= \left(\mathrm{N}_x(1_{\mathrm{X}}, \Theta) * S\left(1_{\mathrm{X}}, x_{\frac{1}{3}}\right)\right) \\
&\quad \vee \left(\mathrm{N}_x\left(x_{\frac{1}{2}}, \Theta\right) * S\left(x_{\frac{1}{2}}, x_{\frac{1}{3}}\right)\right) \vee \left(\mathrm{N}_x\left(x_{\frac{1}{3}}, \Theta\right) * S\left(x_{\frac{1}{3}}, x_{\frac{1}{3}}\right)\right) \\
&= \left(1 * \left(1 \to \tfrac{1}{3}\right)\right) \vee \left(\tfrac{2\Theta}{1+2\Theta} * \left(\tfrac{1}{2} \to \tfrac{1}{3}\right)\right) \\
&\quad \vee \left(\tfrac{3\Theta}{2+3\Theta} * \left(\tfrac{1}{3} \to \tfrac{1}{3}\right)\right) \\
&= \left(1 * \tfrac{1}{3}\right) \vee \left(\tfrac{2\Theta}{1+2\Theta} * \tfrac{5}{6}\right) \vee \left(\tfrac{3\Theta}{2+3\Theta} * 1\right) \\
&= \tfrac{1}{3} \vee \left(\tfrac{2\Theta}{1+2\Theta} - \tfrac{1}{6}\right) \vee \left(\tfrac{3\Theta}{2+3\Theta}\right) \\
&= \tfrac{1}{3} \vee \left(\tfrac{10\Theta - 1}{6+12\Theta}\right) \vee \left(\tfrac{3\Theta}{2+3\Theta}\right)
\end{aligned}$$

Again, the upper LM-Rapprox operator $\overline{\mathrm{N}}$ given by: For $\mathrm{U} = x_{\frac{2}{3}}$

$$\begin{aligned}
\overline{\mathrm{N}}\left(x_{\frac{2}{3}}, \Theta\right)(x) &= \bigwedge_{K \in \mathrm{L}^{\mathrm{X}}} \left(\mathrm{N}_x(K, \Theta) \to T\left(K, x_{\frac{2}{3}}\right)\right) \\
&= \left(\mathrm{N}_x(1_{\mathrm{X}}, \Theta) \to T\left(1_{\mathrm{X}}, x_{\frac{2}{3}}\right)\right) \wedge \left(\mathrm{N}_x\left(x_{\frac{1}{2}}, \Theta\right) \to T\left(x_{\frac{1}{2}}, x_{\frac{2}{3}}\right)\right) \wedge \\
&\quad \left(\mathrm{N}_x\left(x_{\frac{1}{3}}, \Theta\right) \to T\left(x_{\frac{1}{3}}, x_{\frac{2}{3}}\right)\right) \\
&= \left(1 \to \left(1 * \frac{2}{3}\right)\right) \wedge \left(\frac{2\Theta}{1+2\Theta} \to \left(\frac{1}{2} * \frac{2}{3}\right)\right) \wedge \left(\frac{3\Theta}{2+3\Theta} \to \left(\frac{1}{3} * \frac{2}{3}\right)\right) \\
&= \left(1 \to \frac{2}{3}\right) \wedge \left(\frac{2\Theta}{1+2\Theta} \to \frac{1}{6}\right) \wedge \left(\frac{3\Theta}{2+3\Theta} \to 0\right) \\
&= \frac{2}{3} \wedge \frac{7+2\Theta}{6(1+2\Theta)} \wedge \frac{2}{2+3\Theta}
\end{aligned}$$

Remark 2. [29] For all $\Theta \in \mathrm{M}$We observe that every upper (resp., lower) LM-Rapprox operators $\overline{\mathrm{N}}$ (resp., $\underline{\mathrm{N}}$): $\mathrm{L}^{\mathrm{X}} \times \mathrm{M} \to \mathrm{L}^{\mathrm{X}}$ induces upper (resp., lower) approximation operators $\overline{\mathrm{N}}^{\Theta}$(resp., $\underline{\mathrm{N}}^{\Theta}$): $\mathrm{L}^{\mathrm{X}} \to \mathrm{L}^{\mathrm{X}}$ [30, 45] given by $\overline{\mathrm{N}}^{\Theta}(\mathrm{U})(x) = \overline{\mathrm{N}}(\mathrm{U}, \Theta)(x)$(resp., $\underline{\mathrm{N}}^{\Theta}(\mathrm{U})(x) = \underline{\mathrm{N}}(\mathrm{U}, \Theta)(x)$) $\forall\, \mathrm{U} \in \mathrm{L}^{\mathrm{X}}, \forall\, x \in \mathrm{X}$.

Example 4. [29] For $\Theta \in \left\{\frac{2}{3}, \frac{1}{2}\right\}$, the lower LM-Rapprox operator $\underline{\mathrm{N}}$, and the upper LM-Rapprox operator $\overline{\mathrm{N}}$, given in **Example 2**, determines:

(1) The two related lower approximation operators: Let $\mathrm{A} = x_{\frac{1}{3}}$, then

(i) $\underline{\mathrm{N}}^{\left(\frac{2}{3}\right)}\left(x_{\frac{1}{3}}\right)(x) = \underline{\mathrm{N}}\left(x_{\frac{1}{3}}, \frac{2}{3}\right)(x) = \frac{1}{2}$.

(ii) $\underline{\mathrm{N}}^{\left(\frac{1}{2}\right)}\left(x_{\frac{1}{3}}\right)(x) = \underline{\mathrm{N}}\left(x_{\frac{1}{3}}, \frac{1}{2}\right)(x) = \frac{3}{7}$.

(2) The two related upper approximation operators: Let $\mathrm{A} = x_{\frac{2}{3}}$, then

(i) $\overline{\mathrm{N}}^{\left(\frac{2}{3}\right)}(x_{\frac{2}{3}})(x) = \overline{\mathrm{N}}(x_{\frac{2}{3}}, \frac{2}{3})(x) = \frac{1}{2}$.

(ii) $\overline{\mathrm{N}}^{\left(\frac{1}{2}\right)}(x_{\frac{2}{3}})(x) = \overline{\mathrm{N}}(x_{\frac{2}{3}}, \frac{1}{2})(x) = \frac{4}{7}$.

For any $x \in \mathrm{X}$, $\mathrm{U}, \mathrm{V} \in \mathrm{L}^{\mathrm{X}}$ and $\Theta, \Xi \in \mathrm{M}$, an LM-fgns operator $\mathrm{N}: \mathrm{X} \to \mathrm{L}^{\mathrm{L}^{\mathrm{X}} \times \mathrm{M}}$ is called [29]:

(NI) non-increasing, if $\Theta \leq \Xi \Rightarrow \mathrm{N}_x(\mathrm{U}, \Xi) \leq \mathrm{N}_x(\mathrm{U}, \Theta)$.

(SE) serial, if $\mathrm{N}_x(\mathrm{U}, \Theta) \leq \bigvee_{y \in \mathrm{X}} \mathrm{U}(y)$.

(RE) reflexive, if $\mathrm{N}_x(\mathrm{U}, \Theta) \leq \mathrm{U}(x)$.

(UN) unary, if

$$\mathrm{N}_x(\mathrm{U}, \Theta) \otimes \mathrm{N}_x(\mathrm{V}, \Xi) \leq \bigvee_{K \in \mathrm{L}^{\mathrm{X}}} (\mathrm{N}_x(K, \Theta \odot \Xi) \otimes S(K, \mathrm{U} \otimes \mathrm{V})).$$

(TR) transitive, if

$$\mathrm{N}_x(\mathrm{U}, \Theta) \leq \bigvee_{\mathrm{V} \in \mathrm{L}^{\mathrm{X}}} \{\mathrm{N}_x(\mathrm{V}, \Theta) \otimes \bigwedge_{y \in \mathrm{X}} (\mathrm{V}(y) \to \bigvee_{\mathrm{V}_y \in \mathrm{L}^{\mathrm{X}}} (\mathrm{N}_y(\mathrm{V}_y, \Theta) \otimes S(\mathrm{V}_y, \mathrm{U})))\}.$$

The following theorem discusses the properties of the M-level operator of the lower L-fuzzy rough approximation.

Theorem 1.[29] Presume $\mathrm{N}: \mathrm{X} \to \mathrm{L}^{\mathrm{L}^{\mathrm{X}} \times \mathrm{M}}$ is an LM-fgns operator on X. Then the lower LM-Rapprox operator $\underline{\mathrm{N}}$ satisfies the next axioms: $\forall\, \mathrm{U}, \mathrm{V} \in \mathrm{L}^{\mathrm{X}}, \Theta \in \mathrm{M}$,

(1) If L is an st-s-quantale, then $\underline{\mathrm{N}}(\underline{\mathrm{T}}, \Theta) = \underline{\mathrm{T}}$;

(2) $S(\mathrm{U}, \mathrm{V}) \leq S(\underline{\mathrm{N}}(\mathrm{U}, \Theta), \underline{\mathrm{N}}(\mathrm{V}, \Theta))$.

The next theorem presents the properties of the M-level operator of upper L-fuzzy rough approximation.

Theorem 2. [29] Presume $\mathrm{N}: \mathrm{X} \to \mathrm{L}^{\mathrm{L}^{\mathrm{X}} \times \mathrm{M}}$ is an LM-fgns operator on X. Then the upper LM-Rapprox operator $\overline{\mathrm{N}}$ satisfies the next axioms: $\forall\, \mathrm{U}, \mathrm{V} \in \mathrm{L}^{\mathrm{X}}, \Theta \in \mathrm{M}$:

(1) $\overline{\mathrm{N}}(\underline{\perp}, \Theta) = \underline{\perp}$;

(2) $S(\mathrm{U}, \mathrm{V}) \leq S(\overline{\mathrm{N}}(\mathrm{U}, \Theta), \overline{\mathrm{N}}(\mathrm{V}, \Theta))$.

Now, the following theorem discusses some special LM-fgns and the related LM-Rapprox operators.

Theorem 3. [29] Presume $\mathrm{N}: \mathrm{X} \to \mathrm{L}^{\mathrm{L}^{\mathrm{X}} \times \mathrm{M}}$ is an LM-fgns operator on X, for each $\mathrm{U}, \mathrm{V} \in \mathrm{L}^{\mathrm{X}}$ and $\Theta, \Xi \in \mathrm{M}$.

(NI) If N is a non-increasing, then

$\Xi \leq \Theta \Rightarrow \underline{\mathrm{N}}(\mathrm{U}, \Theta) \leq \underline{\mathrm{N}}(\mathrm{U}, \Xi)$ and $\overline{\mathrm{N}}(\mathrm{U}, \Xi) \leq \overline{\mathrm{N}}(\mathrm{U}, \Theta)$.

(SE) (i) N is serial iff $\overline{\mathrm{N}}(\underline{\mathrm{T}}, \Theta) = \underline{\mathrm{T}}$ and $(\mathrm{L}, \leq, \otimes)$ is a st-s-quantale.

(ii) If N is serial, then $\underline{\mathrm{N}}(\underline{\perp}, \Theta) = \underline{\perp}$ and the reverse conclusion held if L satisfy the double negation law.

(RE) Let N be reflexive, then

(i) $\underline{\mathrm{N}}(\mathrm{U}, \Theta) \leq \mathrm{U}$. The reverse is true if $(\mathrm{L}, \leq, \otimes)$ is a st-s-

quantale.
(ii) $\overline{N}(U,\Theta) \geq U$. The reverse is true if L satisfies the double negation law.
(TR) Let N be transitive, then
(i) $\underline{N}(U,\Theta) \leq \underline{N}(\underline{N}(U,\Theta),\Theta)$. The reverse is true if L is a st-s-quantale.
(ii)N is transitive iff $\overline{N}(U,\Theta) \geq \overline{N}(\overline{N}(U,\Theta),\Theta)$ and L satisfy the double negation law.
(UN) Let N be unary, then
$(i)\underline{N}(U \otimes V, \Theta \odot \Xi) \geq \underline{N}(U,\Theta) \otimes \underline{N}(V,\Xi)$. The reverse is true if L is a st-s-quantale.
(ii) $\overline{N}(U \oplus V, \Theta \odot \Xi) \leq \overline{N}(U,\Theta) \oplus \overline{N}(V,\Xi)$, the reverse is true if L is a st-s-quantale, and that with L satisfy the double negation law.

3 Single axiomatic characterizations on lower and upper LM-Rapprox operators

Here we introduce a description of upper and lower LM-Rapprox operators derived from LM-fgns by single axiom.
Theorem 4. Presume $h: L^X \times M \to L^X$ is an operator, then there exists an LM-fgns operator $N: X \to L^{L^X \times M}$ such that: for each $U,V \in L^X$ and $\Theta \in M$.
(1) $h = \underline{N}$ iff h has the following property:
$(SNGL_1)$ $S(U,V) \leq S(h(U,\Theta),h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x)$.
(2) $h = \overline{N}$ iff h has the following property:
$(SNUL_1)$ $S(U,V) \leq S(h(U,\Theta),h(V,\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x)$.

Proof. (1) Suppose that $N: X \to L^{L^X \times M}$ is an LM-fgns operator on X and $h = \underline{N}$. Then by **Theorem 1** we have that h satisfies (1) and (2). Next, we verify the condition $(SNGL_1)$. For any $U,V \in L^X$ and $\Theta \in M$, we have

$$\begin{aligned} S(U,V) &\leq S(h(U,\Theta),h(V,\Theta)) \\ &= S\big(h(U,\Theta),h(V,\Theta)\big) \otimes \top \\ &= S(h(U,\Theta),h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

Thus, h satisfies $S(U,V) \leq S(h(U,\Theta),h(V,\Theta)) \otimes h(\underline{\top},\Theta)$.
Conversely, suppose that $h: L^X \times M \to L^X$ is an operator satisfying $(SNGL_1)$. For each $\Theta \in M$, we put $U = V$, then we find

$$\top \leq \top \otimes h(\underline{\top},\Theta)(x)$$

therefore $\top \leq h(\underline{\top},\Theta)(x)$ for each $x \in X$. Hence, $h(\underline{\top},\Theta) = \underline{\top}$. Further, it follows from $(SNGL_1)$, we have $S(U,V) \leq S(h(U,\Theta),h(V,\Theta))$. Then by **Theorem 6.32** in [46], we know there exists an LM-fgns operator on X such that $h = \underline{N}$.
(2) This is similar to proving (1) by utilizing dual properties.

3.1 Characterizing lower and upper LM-Rapprox operators generated by each kind of LM-fgns with single axioms

In this subsection, we present single axiomatic characterizations of lower and upper LM-Rapprox operators w.r.t non-increasing, unary, reflexive, serial, and transitive LM-fgns, respectively.
Theorem 5. Let $h: L^X \times M \to L^X$ be an operator, then there exists a non-increasing LM-fgns operator $N: X \to L^{L^X \times M}$ such that: for any $U,V \in L^X$ and $\Theta,\Xi \in M$
(1) $h = \underline{N}$ iff the operator h satisfies $(SNGL_{NI})$:
$S(U,V) \leq S(h(U,\Theta),h(V,\Theta) \wedge h(V,\Xi)) \otimes h(\underline{\top},\Theta)(x)$
whenever $\Xi \leq \Theta$.
(2)$h = \overline{N}$ iff the operator h satisfies $(SNGU_{NI})$:
$S(U,V) \leq S(h(U,\Theta) \vee h(U,\Xi),h(V,\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x)$
whenever $\Xi \leq \Theta$.

Proof. (1) Suppose that $N: X \to L^{L^X \times M}$ is a non-increasing LM-fgns operator on X and $h = \underline{N}$. Then it follows from **Theorem 3** (NI), that $h(V,\Theta) \leq h(V,\Xi)$ whenever $\Xi \leq \Theta$. Then through **Theorem 4**, for each $U,V \in L^X, \Theta,\Xi \in M$, and $x \in X$, we get

$$\begin{aligned} S(U,V) &\leq S(h(U,\Theta),h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(U,\Theta),h(V,\Theta) \wedge h(V,\Xi)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

Conversely, assume that $h: L^X \times M \to L^X$ is an operator satisfying $(SNGL_{NI})$. By taking $U = V$, for each $U,V \in L^X$, and $\Theta,\Xi \in M$ with $\Xi \leq \Theta$, we get that

$$\top \leq S(h(V,\Theta),h(V,\Theta) \wedge h(V,\Xi)) \otimes h(\underline{\top},\Theta)(x)$$

This means that

$$\begin{aligned} &\top \leq S(h(V,\Theta),h(V,\Theta) \wedge h(V,\Xi)) \\ &\Leftrightarrow h(V,\Theta) \leq h(V,\Theta) \wedge h(V,\Xi) \\ &\Leftrightarrow h(V,\Theta) = h(V,\Theta) \wedge h(V,\Xi) \\ &\Leftrightarrow h(V,\Theta) \leq h(V,\Xi). \end{aligned}$$

Then it follows from $(SNGL_{NI})$ that

$$\begin{aligned} S(U,V) &\leq S(h(U,\Theta),h(V,\Theta) \wedge h(V,\Xi)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(U,\Theta),h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

So through **Theorem 4**, there exists an LM-fgns operator on X such that $h = \underline{N}$. Further, we have

$$\underline{N}(V,\Theta) = h(V,\Theta) \leq h(V,\Xi) = \underline{N}(V,\Xi) \text{ whenever } \Xi \leq \Theta$$

This implies that N is non-increasing.
(2) This is similar to proving (1) by utilizing dual properties.
Corollary 1. Let $h: L^X \times M \to L^X$ be an operator, then there exists a non-increasing LM-fgns operator $N: X \to L^{L^X \times M}$ such that: for any $U,V,W \in L^X$, and $\Theta,\Xi \in M$.
(1) $h = \underline{N}$ iff the operator h satisfies $(SNGLL_{NI})$:
$S(U,V) \leq S\big(h(U,\Theta),h(V,\Theta) \wedge h(W,\Xi)\big) \otimes h\big(\underline{\top},\Theta\big)(x)$
whenever with $V \leq W$, and $\Xi \leq \Theta$.
(2) $h = \overline{N}$ iff the operator h satisfies $(SNGUU_{NI})$:
$S(U,V) \leq S(h(U,\Theta) \vee h(W,\Xi),h(V,\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x)$
whenever with $W \leq U$, and $\Xi \leq \Theta$.
Theorem 6. Let $h: L^X \times M \to L^X$ be an operator and L satisfying the double negation law, then there exists a serial LM-fgns operator $N: X \to L^{L^X \times M}$ such that: for each $U,V \in L^X$ and $\Theta \in M$.
(1)$h = \underline{N}$ iff the operator h satisfies $(SNGL_{SE})$ as follows
$S(U,V) \leq S(h(U,\Theta),h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x)$.
(2) $h = \overline{N}$ iff the operator h satisfies $(SNGU_{SE})$ as follows
$S(U,V) \leq S(h(U,\Theta),h(V,\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes h(\underline{\top},\Theta)(x)$.

Proof. (1) Suppose that $N: X \to L^{L^X \times M}$ is a serial LM-fgns operator on X and $h = \underline{N}$. Then by **Theorem 3** (SE), we have $h(\underline{\perp},\Theta) = \underline{\perp}$, i.e., $\neg h(\underline{\perp},\Theta) = \underline{\top}$. Then by **Theorem 4**, for each $U,V \in L^X, \Theta \in M$, and $x \in X$, we have

$$\begin{aligned} S(U,V) &\leq S(h(U,\Theta),h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S\big(h(U,\Theta),h(V,\Theta)\big) \otimes h\big(\underline{\top},\Theta\big)(x) \otimes \top \\ &= S(h(U,\Theta),h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x). \end{aligned}$$

Hence, h satisfy:

$$S(U,V) \leq S(h(U,\Theta),h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x).$$

Conversely, assume that $h: L^X \times M \to L^X$ is an operator satisfying $(SNGL_{SE})$. For each $\Theta \in M$, we put $U = V$, then we have

$$\top \leq \top \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x)$$

This means that $\top \leq \neg h(\underline{\perp}, \Theta)(x)$ for each $x \in \mathrm{X}$, and therefore $\neg h(\underline{\perp}, \Theta) = \underline{\top}$, i.e., $h(\underline{\perp}, \Theta) = \underline{\perp}$. From $(SNGL_{SE})$ it follows that

$$\begin{aligned} S(\mathrm{U},\mathrm{V}) &\leq S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x) \\ &= S\big(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)\big) \otimes h\big(\underline{\top},\Theta\big)(x) \otimes \top \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

Then there exists an LM-fgns operator N on X with $h = \underline{\mathrm{N}}$ by **Theorem 4**. Moreover, we have

$$\underline{\mathrm{N}}(\underline{\perp}, \Theta) = h(\underline{\perp}, \Theta) = \underline{\perp}.$$

This lead to N is a serial.

(2) This is similar to proving (1) by utilizing dual properties.

Theorem 7.Presume $h\colon \mathrm{L^X} \times \mathrm{M} \to \mathrm{L^X}$ is an operator, then there exists a reflexive LM-fgns operator $\mathrm{N}\colon \mathrm{X} \to \mathrm{L}^{\mathrm{L^X}\times \mathrm{M}}$ such that: for each $\mathrm{U},\mathrm{V} \in \mathrm{L^X}$, and $\Theta \in \mathrm{M}$.

(1) $h = \underline{\mathrm{N}}$ iff the operator h satisfies $(SNGL_{RE})$ i.e., $S(\mathrm{U},\mathrm{V}) \leq S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V}) \otimes h(\underline{\top},\Theta)(x)$.

(2) $h = \overline{\mathrm{N}}$ iff the operator h satisfies $(SNGU_{RE})$ i.e., $S(\mathrm{U},\mathrm{V}) \leq S(h(\mathrm{U},\Theta) \vee \mathrm{U}, h(\mathrm{V},\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x)$.

Proof. (1) Suppose that $\mathrm{N}\colon \mathrm{X} \to \mathrm{L}^{\mathrm{L^X}\times \mathrm{M}}$ is a reflexive LM-fgns operator on X and $h = \underline{\mathrm{N}}$. Then it follows from **Theorem 3** (RE), that $h(\mathrm{V},\Theta) \leq \mathrm{V}$. Then through **Theorem 4**, for each $\mathrm{U},\mathrm{V} \in \mathrm{L^X}, \Theta \in \mathrm{M}$, and $x \in \mathrm{X}$, we get

$$\begin{aligned} S(\mathrm{U},\mathrm{V}) &\leq S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V}) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

Conversely, suppose that $h\colon \mathrm{L^X} \times \mathrm{M} \to \mathrm{L^X}$ is an operator satisfying $(SNGL_{RE})$. By taking $\mathrm{U} = \mathrm{V}$, for each $\Theta \in \mathrm{M}$, we get that

$$\top \leq S(h(\mathrm{V},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V}) \otimes h(\underline{\top},\Theta)(x)$$

This means that

$$\begin{aligned} &\top \leq S(h(\mathrm{V},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V}) \\ &\Leftrightarrow h(\mathrm{V},\Theta) \leq h(\mathrm{V},\Theta) \wedge \mathrm{V} \\ &\Leftrightarrow h(\mathrm{V},\Theta) = h(\mathrm{V},\Theta) \wedge \mathrm{V} \\ &\Leftrightarrow h(\mathrm{V},\Theta) \leq \mathrm{V}. \end{aligned}$$

Then it follows from $(SNGL_{RE})$ that

$$\begin{aligned} S(\mathrm{U},\mathrm{V}) &\leq S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V}) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

So through **Theorem 4**, there exists an LM-fgns operator on X such that $h = \underline{\mathrm{N}}$. Further,we have

$$\underline{\mathrm{N}}(\mathrm{V},\Theta) = h(\mathrm{V},\Theta) \leq \mathrm{V}.$$

This implies that N is reflexive.

(2) This is similar to proving (1) by utilizing dual properties.

Theorem 8. Presume $h\colon \mathrm{L^X} \times \mathrm{M} \to \mathrm{L^X}$ is an operator, then there exists an unary LM-fgns operator $\mathrm{N}\colon \mathrm{X} \to \mathrm{L}^{\mathrm{L^X}\times \mathrm{M}}$ such that: for each $\mathrm{U},\mathrm{V},\mathrm{C},\mathrm{D} \in \mathrm{L^X}$, and $\Theta, \Xi \in \mathrm{M}$.

(1) $h = \underline{\mathrm{N}}$ iff the operator h satisfies $(SNGL_{UN})$ i.e., $S(\mathrm{U},\mathrm{V}) \leq S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C} \otimes \mathrm{D}, \Theta \odot \Xi))$,

(2) $h = \overline{\mathrm{N}}$ iff the operator h satisfies $(SNGU_{UN})$ i.e., $S(\mathrm{U},\mathrm{V}) \leq S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes S(h(\mathrm{C} \oplus \mathrm{D}, \Theta \odot \Xi), h(\mathrm{C},\Theta) \oplus h(\mathrm{D},\Xi))$.

Proof. (1) Suppose that $\mathrm{N}\colon \mathrm{X} \to \mathrm{L}^{\mathrm{L^X}\times \mathrm{M}}$ is an unary LM-fgns operator on X and $h = \underline{\mathrm{N}}$. Then, we have from **Theorem 3** (UN), that $h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi) \leq h(\mathrm{C} \otimes \mathrm{D}, \Theta \odot \Xi)$, i.e.,

$$S\big(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C} \otimes \mathrm{D}, \Theta \odot \Xi)\big) = \top.$$

Then through **Theorem 4**, it follows, for each $\mathrm{U},\mathrm{V},\mathrm{C},\mathrm{D} \in \mathrm{L^X}, \Theta, \Xi \in \mathrm{M}$, and $x \in \mathrm{X}$, that:

$$\begin{aligned} S(\mathrm{U},\mathrm{V}) &\leq S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S\big(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)\big) \otimes h\big(\underline{\top},\Theta\big)(x) \otimes \top \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &\quad \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C} \otimes \mathrm{D}, \Theta \odot \Xi)). \end{aligned}$$

Conversely, assume that $h\colon \mathrm{L^X} \times \mathrm{M} \to \mathrm{L^X}$ is an operator satisfying $(SNGL_{UN})$. By taking $\mathrm{U} = \mathrm{V}$, for each $\Theta \in \mathrm{M}$, we get that

$$\top \leq \top \otimes h(\underline{\top},\Theta)(x) \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C} \otimes \mathrm{D}, \Theta \odot \Xi))$$

Therefore

$$\begin{aligned} &\top \leq S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C} \otimes \mathrm{D}, \Theta \odot \Xi)) \\ &\Leftrightarrow \top = S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C} \otimes \mathrm{D}, \Theta \odot \Xi)) \\ &\Leftrightarrow h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi) \leq h(\mathrm{C} \otimes \mathrm{D}, \Theta \odot \Xi). \end{aligned}$$

Then it follows from $(SNGL_{UN})$ that

$$\begin{aligned} S(\mathrm{U},\mathrm{V}) &\leq S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \\ &\quad S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C} \otimes \mathrm{D}, \Theta \odot \Xi)) \\ &= S\big(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)\big) \otimes h\big(\underline{\top},\Theta\big)(x) \otimes \top \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

Then from **Theorem 4**, there exists an LM-fgns operator on X such that $h = \underline{\mathrm{N}}$. Furthermore, we get

$$\underline{\mathrm{N}}(\mathrm{C} \otimes \mathrm{D}, \Theta \odot \Xi) = h(\mathrm{C} \otimes \mathrm{D}, \Theta \odot \Xi) \geq h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi) = \underline{\mathrm{N}}(\mathrm{C},\Theta) \otimes \underline{\mathrm{N}}(\mathrm{D},\Xi),$$

which means that N is unary.

(2) This is similar to proving (1) by utilizing dual properties.

Theorem 9. Presume $h\colon \mathrm{L^X} \times \mathrm{M} \to \mathrm{L^X}$ is an operator, then there exists a transitive LM-fgns operator $\mathrm{N}\colon \mathrm{X} \to \mathrm{L}^{\mathrm{L^X}\times \mathrm{M}}$ such that: for each $\mathrm{U},\mathrm{V} \in \mathrm{L^X}$ and $\Theta \in \mathrm{M}$.

(1) $h = \underline{\mathrm{N}}$ iff the operator h satisfies $(SNGL_{TR})$ i.e., $S(\mathrm{U},\mathrm{V}) \leq S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x)$,

(2) $h = \overline{\mathrm{N}}$ iff the operator h satisfies $(SNGU_{TR})$ i.e., $S(\mathrm{U},\mathrm{V}) \leq S(h(\mathrm{U},\Theta) \vee h(h(\mathrm{U},\Theta),\Theta), h(\mathrm{V},\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x)$.

Proof. (1) Suppose that $\mathrm{N}\colon \mathrm{X} \to \mathrm{L}^{\mathrm{L^X}\times \mathrm{M}}$ is a transitive LM-fgns operator on X and $h = \underline{\mathrm{N}}$. then it follows from **Theorem 3** (TR), that $h(\mathrm{V},\Theta) \leq h(h(\mathrm{V},\Theta),\Theta)$. By **Theorem 4**, it follows, for each $\mathrm{U},\mathrm{V} \in \mathrm{L^X}, \Theta \in \mathrm{M}$, and $x \in \mathrm{X}$, that

$$\begin{aligned} S(U,V) &\leq S(h(U,\Theta), h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

Conversely, assume that $h\colon \mathrm{L^X} \times \mathrm{M} \to \mathrm{L^X}$ is an operator satisfying $(SNGL_{TR})$. For each $\Theta \in \mathrm{M}$, if we put $\mathrm{U} = \mathrm{V}$ we have

$$\top \leq S(h(\mathrm{V},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x).$$

Therefore

$$\begin{aligned} &\top \leq S(h(\mathrm{V},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \\ &\Leftrightarrow h(\mathrm{V},\Theta) \leq h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta) \\ &\Leftrightarrow h(\mathrm{V},\Theta) = h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta) \\ &\Leftrightarrow h(\mathrm{V},\Theta) \leq h(h(\mathrm{V},\Theta),\Theta). \end{aligned}$$

Then it follows from $(SNGL_{TR})$ that

$$\begin{aligned} S(\mathrm{U},\mathrm{V}) &\leq S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

Then from **Theorem 4**, there exists an LM-fgns operator on X such that $h = \underline{\mathrm{N}}$. Further, there is

$$\underline{\mathrm{N}}(\mathrm{V},\Theta) = h(\mathrm{V},\Theta) \leq h(h(\mathrm{V},\Theta),\Theta) = \underline{\mathrm{N}}(\underline{\mathrm{N}}(\mathrm{V},\Theta),\Theta),$$

which means that N is a transitive.

(2) This is similar to proving (1) by utilizing dual properties.

3.2 Characterizing lower and upper LM-Rapprox operators induced by compositions of LM-fgns

Now, we will discuss single axiomatic characterizations of upper and lower LM-Rapprox operators w.r.t any compositions of non-increasing, unary, reflexive, serial, and transitive LM-fgns.

Theorem 10. Presume L is satisfied the double negation law and $h: L^X \times M \to L^X$ is an operator. Then there exists a non-increasing and reflexive LM-fgns operator N on X such that: for each $U, V \in L^X$, and $\Theta, \Xi \in M$.

(1)$\underline{N} = h$ iff the operator h satisfies $(NRGL)$ i.e.,
$S(U,V) \leq S(h(U,\Theta), h(V,\Theta) \wedge h(V,\Xi) \wedge V) \otimes h(\underline{\top},\Theta)(x)$
whenever $\Xi \leq \Theta$.

(2) $\overline{N} = h$ iff the operator h satisfies $(NRGU)$ i.e., $S(U,V) \leq S(h(U,\Theta) \vee h(U,\Xi) \vee U, h(V,\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x)$ whenever $\Xi \leq \Theta$.

Proof. (1) Suppose that $N: X \to L^{L^X \times M}$ is a non-increasing and reflexive LM-fgns operator on X and $\underline{N} = h$. Then it follows from the items (NI) and (RE) of **Theorem 3** that $h(V,\Theta) \leq V$ and $h(V,\Theta) \leq h(V,\Xi)$ for each $V \in L^X$ and $\Theta, \Xi \in M$ with $\Xi \leq \Theta$. Then by **Theorem 4**, we have

$$S(U,V) \leq S(h(U,\Theta), h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x)$$
$$= S(h(U,\Theta), h(V,\Theta) \wedge h(V,\Xi) \wedge V) \otimes h(\underline{\top},\Theta)(x).$$

Hence, h satisfies $(NRGL)$.

Conversely, suppose that $h: L^X \times M \to L^X$ is an operator satisfying $(NRGL)$. If we put $U = V$, for each $\Theta, \Xi \in M$ with $\Xi \leq \Theta$, then we have

$$\top \leq S(h(V,\Theta), h(V,\Theta) \wedge h(V,\Xi) \wedge V) \otimes h(\underline{\top},\Theta)(x)$$

Therefore

$$\top \leq S(h(V,\Theta), h(V,\Theta) \wedge h(V,\Xi) \wedge V)$$
$$\Leftrightarrow h(V,\Theta) \leq h(V,\Theta) \wedge h(V,\Xi) \wedge V$$
$$\Rightarrow h(V,\Theta) \leq h(V,\Xi) \leq V$$

Then it follows from $(NRGL)$ that

$$S(U,V) \leq S(h(U,\Theta), h(V,\Theta) \wedge h(V,\Xi) \wedge V) \otimes h(\underline{\top},\Theta)(x)$$
$$= S(h(U,\Theta), h(V,\Theta) \wedge h(V,\Xi)) \otimes h(\underline{\top},\Theta)(x).$$

Through **Theorem 5**, there exists a non-increasing LM-fgns operator on X such that $\underline{N} = h$. Further, we have $\underline{N}(V,\Theta) = h(V,\Theta) \leq V, \ \forall \ V \in L^X and \ \Theta \in M$,
which means that N is reflexive.

(2)This is similar to proving (1) by utilizing dual properties.

Theorem 11. Presume L is satisfied the double negation law and $h: L^X \times M \to L^X$ is an operator, then there exists a non-increasing and unary LM-fgns operator N on X such that: for each $U, V, C, D \in L^X$, and $\Theta, \Xi \in M$.

(1) $\underline{N} = h$ iff the operator h satisfies $(NUGL)$ i.e., $S(U,V) \leq S(h(U,\Theta), h(V,\Theta) \wedge h(V,\Xi)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi))$ whenever $\Xi \leq \Theta$.

(2)$\overline{N} = h$ iff the operator h satisfies $(NUGU)$ i.e., $S(U,V) \leq S(h(U,\Theta) \vee h(U,\Xi), h(V,\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes S(h(C \oplus D, \Theta \odot \Xi), h(C,\Theta) \oplus h(D,\Xi))$
whenever $\Xi \leq \Theta$.

Proof. The proof is carried out in the same style as the proof of the previous result.

Theorem 12. Presume L is satisfied the double negation law and $h: L^X \times M \to L^X$ is an operator. Then there exists a serial and transitive LM-fgns operator N on X such that: for each $U, V \in L^X$, and $\Theta \in M$.

(1) $\underline{N} = h$ iff the operator h satisfies $(STGL)$:
$S(U,V) \leq S\big(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)\big) \otimes h\big(\underline{\top},\Theta\big)(x) \otimes \neg h(\underline{\perp},\Theta)(x)$.

(2) $\overline{N} = h$ iff the operator h satisfies $(STGU)$:
$S(U,V) \leq S\big(h(U,\Theta) \vee h(h(U,\Theta),\Theta), h(V,\Theta)\big) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes h(\underline{\top},\Theta)(x)$

Proof. (1) Suppose that $N: X \to L^{L^X \times M}$ is a serial and transitive LM-fgns operator N on X and $\underline{N} = h$. Then it follows from **Theorem 3** (SE) and (TR) that $h(\underline{\perp},\Theta) = \underline{\perp}$, i.e., $\neg h(\underline{\perp},\Theta) = \underline{\top}$ and $h(h(V,\Theta),\Theta) \geq h(V,\Theta)$ for each $V \in L^X$ and $\Theta \in M$. By **Theorem 4**, we have that

$$S(U,V) \leq S(h(U,\Theta), h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x)$$
$$= S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h\big(\underline{\top},\Theta\big)(x) \otimes \top$$
$$= S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x)$$

Thus, h satisfies $(STGL)$. Conversely, assume that $h: L^X \times M \to L^X$ is an operator satisfying $(STGL)$. For each $\Theta \in M$, if we put $U = V$, we have that $\top \leq S(h(V,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x)$. This means that $\top \leq \neg h(\underline{\perp},\Theta)(x)$ for each $x \in X$. Hence $\neg h(\underline{\perp},\Theta) = \underline{\top}$, i.e., $h(\underline{\perp},\Theta) = \underline{\perp}$. From $(STGL)$ it follows that

$$S(U,V) \leq S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x)$$
$$= S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h\big(\underline{\top},\Theta\big)(x) \otimes \top$$
$$= S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x).$$

So through **Theorem 9,** there exists a transitive LM-fgns operator on X such that $\underline{N} = h$. Further, we have

$$\underline{N}(\underline{\perp},\Theta) = h(\underline{\perp},\Theta) = \underline{\perp},$$

which means that N is serial.

(2) This is similar to proving (1) by utilizing dual properties.

Theorem 13. Presume L is satisfied the double negation law and $h: L^X \times M \to L^X$ is an operator. Then there exists a serial and unary LM-fgns operator N on X such that: for each $U, V, C, D \in L^X$, and $\Theta, \Xi \in M$.

(1) $\underline{N} = h$ iff the operator h satisfies $(SUGL)$:
$S(U,V) \leq S\big(h(U,\Theta), h(V,\Theta)\big) \otimes h\big(\underline{\top},\Theta\big)(x) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi))$

(2) $\overline{N} = h$ iff the operator h satisfies $(SUGU)$:
$S(U,V) \leq S(h(U,\Theta), h(V,\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C \oplus D, \Theta \odot \Xi), h(C,\Theta) \oplus h(D,\Xi))$.

Proof. The proof is carried out in the same style as the proof of the previous result.

Theorem 14. Let $h: L^X \times M \to L^X$ be an operator. Then there exists a reflexive and unary LM-fgns operator N on X such that: for each $U, V, C, D \in L^X$, and $\Theta, \Xi \in M$.

(1) $\underline{N} = h$ iff the operator h satisfies $(RUGL)$:
$S(U,V) \leq S(h(U,\Theta), h(V,\Theta) \wedge V) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi))$.

(2) $\overline{N} = h$ iff the operator h satisfies $(RUGU)$:
$S(U,V) \leq S(h(U,\Theta) \vee U, h(V,\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes$

$S(h(C \oplus D, \Theta \odot \Xi), h(C,\Theta) \oplus h(D,\Xi))$.

Proof. (1) Suppose that $N: X \to L^{L^X \times M}$ is a reflexive and unary LM-fgns operator on X and $\underline{N} = h$. Then it follows from the items (RE) and (UN) of **Theorem 3** that $h(V,\Theta) \leq V$ and $h(C,\Theta) \otimes h(D,\Xi) \leq h(C \otimes D, \Theta \odot \Xi)$ for each $U, V, C, D \in L^X$ and $\Theta, \Xi \in M$. Then by **Theorem 4**, we get that

$$\begin{aligned} S(U,V) &\leq S(h(U,\Theta), h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(U,\Theta), h(V,\Theta) \wedge V) \otimes h(\underline{\top},\Theta)(x) \otimes \top \\ &= S(h(U,\Theta), h(V,\Theta) \wedge V) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi)). \end{aligned}$$

Thus, h satisfies $(RUGL)$.

Conversely, suppose that $h: L^X \times M \to L^X$ is an operator satisfying $(RUGL)$. For each $\Theta, \Xi \in M$, if we take $U = V$, then we have

$$\top \leq S(h(V,\Theta), h(V,\Theta) \wedge V) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi))$$

Therefore

$$\begin{aligned} \top \leq S(h(V,\Theta), h(V,\Theta) \wedge V) &\Leftrightarrow h(V,\Theta) \leq h(V,\Theta) \wedge V \\ &\Leftrightarrow h(V,\Theta) = h(V,\Theta) \wedge V \\ &\Leftrightarrow h(V,\Theta) \leq V. \end{aligned}$$

Then it follows from $(RUGL)$ that

$$\begin{aligned} S(U,V) &\leq S(h(U,\Theta), h(V,\Theta) \wedge V) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi)) \\ &= S(h(U,\Theta), h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi)). \end{aligned}$$

So through **Theorem 8**, there exists an unary LM-fgns operator on X such that $\underline{N} = h$. Further, we have

$$\underline{N}(V,\Theta) = h(V,\Theta) \leq V, \quad \forall\ V \in L^X \mathit{and}\ \ \Theta \in M.$$

This implies that N is reflexive.

(2) This is similar to proving (1) by utilizing dual properties.

Theorem 15. Let $h: L^X \times M \to L^X$ be an operator. Then there exists a reflexive and transitive LM-fgns operator N on X such that: for each $U, V \in L^X$, and $\Theta \in M$.

(1) $\underline{N} = h$ iff the operator h satisfies $(RTGL)$:

$$S(U,V) \leq S(h(U,\Theta), h(V,\Theta) \wedge V \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x)$$

(2) $\overline{N} = h$ iff the operator h satisfies $(RTGU)$:

$$S(U,V) \leq S(h(U,\Theta) \vee U \vee h(h(U,\Theta),\Theta), h(V,\Theta)) \otimes \neg h(\underline{\bot},\Theta)(x).$$

Proof. (1) Suppose that $N: X \to L^{L^X \times M}$ is a reflexive and transitive LM-fgns operator on X and $\underline{N} = h$. Then it follows from the elements (RE) and (TR) of **Theorem 3** that $h(V,\Theta) \leq V$ and $h(V,\Theta) \leq h(h(V,\Theta),\Theta)$ for each $V \in L^X$ and $\Theta \in M$. Then by **Theorem 4**, we have

$$\begin{aligned} S(U,V) &\leq S(h(U,\Theta), h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(U,\Theta), h(V,\Theta) \wedge V \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

Hence, h satisfies $(RTGL)$.

Conversely, assume that $h: L^X \times M \to L^X$ is an operator satisfying $(RTGL)$. If we put $U = V$, for each $\Theta \in M$, then we have

$$\top \leq S(h(V,\Theta), h(V,\Theta) \wedge V \wedge h(h(V,\Theta),\Theta)) \otimes (\underline{\top},\Theta)(x)$$

Therefore

$$\begin{aligned} &\top \leq S(h(V,\Theta), h(V,\Theta) \wedge V \wedge h(h(V,\Theta),\Theta)) \\ &\Leftrightarrow h(V,\Theta) \leq h(V,\Theta) \wedge V \wedge h(h(V,\Theta),\Theta) \\ &\Rightarrow h(V,\Theta) \leq V \end{aligned}$$

Then it follows from $(RTGL)$ that

$$\begin{aligned} S(U,V) &\leq S(h(U,\Theta), h(V,\Theta) \wedge V \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

Through **Theorem 9**, there exists a transitive LM-fgns operator on X such that $\underline{N} = h$. Further, we have

$$\underline{N}(V,\Theta) = h(V,\Theta) \leq V, \quad \forall\ V \in L^X \mathit{and}\ \ \Theta \in M,$$

which means that N is reflexive.

(2) This is similar to proving (1) by utilizing dual properties.

Theorem 16. Let $h: L^X \times M \to L^X$ be an operator. Then there exists an unary and transitive LM-fgns operator N on X such that: for each $U, V, C, D \in L^X$, and $\Theta, \Xi \in M$.

(1) $\underline{N} = h$ iff the operator h satisfies the axiom $(UTGL)$:

$$S(U,V) \leq S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi)).$$

(2) $\overline{N} = h$ iff the operator h satisfies the axiom $(UTGU)$:

$$S(U,V) \leq S(h(U,\Theta) \vee h(h(U,\Theta),\Theta), h(V,\Theta)) \otimes \neg h(\underline{\bot},\Theta)(x) \otimes S(h(C \oplus D, \Theta \odot \Xi), h(C,\Theta) \oplus h(D,\Xi)).$$

Proof. (1) Suppose that $N: X \to L^{L^X \times M}$ is an unary and transitive LM-fgns operator on X and $\underline{N} = h$. Then it follows from the items (UN) and (TR) of **Theorem 3** that $h(C,\Theta) \otimes h(D,\Xi) \leq h(C \otimes D, \Theta \odot \Xi)$ and $h(V,\Theta) \leq h(h(V,\Theta),\Theta)$ for each $U, V, C, D \in L^X$ and $\Theta, \Xi \in M$. Then by **Theorem 4**, we get that

$$\begin{aligned} S(U,V) &\leq S(h(U,\Theta), h(V,\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \top \\ &= S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi)) \end{aligned}$$

So, h satisfies $(UTGL)$.

Conversely, assume that $h: L^X \times M \to L^X$ is an operator satisfying $(UTGL)$. Taking $U = V$, for each $\Theta, \Xi \in M$, we have

$$\top \leq S(h(V,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi)),$$

It follows that

$$\begin{aligned} &\top \leq S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi)) \\ &\Leftrightarrow \top = S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi)) \\ &\Leftrightarrow h(C,\Theta) \otimes h(D,\Xi) \leq h(C \otimes D, \Theta \odot \Xi) \end{aligned}$$

Then it follows from $(UTGL)$ that

$$\begin{aligned} S(U,V) &\leq S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi)) \\ &= S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

From **Theorem 9**, there exists a transitive LM-fgns operator N on X such that $\underline{N} = h$. Furthermore, we get

$$\underline{N}(C,\Theta) \otimes \underline{N}(D,\Xi) = h(C,\Theta) \otimes h(D,\Xi) \leq h(C \otimes D, \Theta \odot \Xi) = \underline{N}(C \otimes D, \Theta \odot \Xi), \quad \forall\ C, D \in L^X \mathit{and}\ \ \Theta, \Xi \in M.$$

Therefore, N is unary.

(2) This is similar to proving (1) by utilizing dual properties.

Theorem 17. Let L be satisfied the double negation law and $h: L^X \times M \to L^X$ be an operator, then there exists a serial, unary and transitive LM-fgns operator N on X such that: for each $U, V, C, D \in L^X$, and $\Theta, \Xi \in M$.

(1) $\underline{N} = h$ iff the operator h satisfies the axiom $(SUTGL)$:

$S(\mathrm{U},\mathrm{V}) \le S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes (\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi))$.

(2) $\overline{\mathrm{N}} = h$ iff the operator h satisfies the axiom $(SUTGU)$:

$$S(\mathrm{U},\mathrm{V}) \le S(h(\mathrm{U},\Theta) \vee h(h(\mathrm{U},\Theta),\Theta), h(\mathrm{V},\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(\mathrm{C}\oplus \mathrm{D},\Theta\odot\Xi), h(\mathrm{C},\Theta) \oplus h(\mathrm{D},\Xi)).$$

Proof. (1) Assume that $\mathrm{N}: \mathrm{X} \to \mathrm{L}^{\mathrm{L}^{\mathrm{X}}\times \mathrm{M}}$ is a serial, unary and transitive LM-fgns operator N on X and $\underline{\mathrm{N}} = h$. Then it follows from **Theorem 3** (SE), (UN) and (TR) that $h(\underline{\perp},\Theta) = \underline{\perp}$, i.e., $\neg h(\underline{\perp},\Theta) = \underline{\top}$, $h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi) \le h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)$ and $h(\mathrm{V},\Theta) \le h(h(\mathrm{V},\Theta),\Theta)$ for each $\mathrm{U},\mathrm{V},\mathrm{C},\mathrm{D} \in \mathrm{L}^{\mathrm{X}}$ and $\Theta,\Xi \in \mathrm{M}$. Then by **Theorem 4**, we have

$$\begin{aligned} S(\mathrm{U},\mathrm{V}) &\le S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \top \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x) \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes \top \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)) \end{aligned}$$

Hence, h satisfies $(SUTGL)$.

Conversely, assume that the operator $h: \mathrm{L}^{\mathrm{X}} \times \mathrm{M} \to \mathrm{L}^{\mathrm{X}}$ is satisfying the axiom $(SUTGL)$. For each $\Theta,\Xi \in \mathrm{M}$, if we put $\mathrm{U} = \mathrm{V}$, then we have

$$\top \le S(h(\mathrm{V},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes S([h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi)], [h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)])$$

Therefore

$$\begin{aligned} &\top \le S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)) \\ &\Leftrightarrow \top = S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)) \\ &\Leftrightarrow h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi) \le h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi) \end{aligned}$$

Then it follows from $(SUTGL)$ that

$$\begin{aligned} S(\mathrm{U},\mathrm{V}) &\le S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{T},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)) \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{T},\Theta)(x) \otimes \neg h(\underline{\perp},\Theta)(x). \end{aligned}$$

By **Theorem 12**, there exists a serial and transitive LM-fgns operator on X such that $\underline{\mathrm{N}} = h$. Further, we have

$\underline{\mathrm{N}}(\mathrm{C},\Theta) \otimes \underline{\mathrm{N}}(\mathrm{D},\Xi) = h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi) \le h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi) = \underline{\mathrm{N}}(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)$, $\forall\ \mathrm{C},\mathrm{D} \in \mathrm{L}^{\mathrm{X}}$ *and* $\Theta,\Xi \in \mathrm{M}$.

Therefore, N is unary.

(2) This is similar to proving (1) by utilizing dual properties.

Theorem 18 Let $h: \mathrm{L}^{\mathrm{X}} \times \mathrm{M} \to \mathrm{L}^{\mathrm{X}}$ be an operator. Then there exists a reflexive, unary and non-increasing LM-fgns operator on X such that: for each $\mathrm{U},\mathrm{V},\mathrm{C},\mathrm{D} \in \mathrm{L}^{\mathrm{X}}$, and $\Theta,\Xi \in \mathrm{M}$.

(1) $\underline{\mathrm{N}} = h$ iff the operator h satisfies the axiom $(RUNGL)$:

$S(\mathrm{U},\mathrm{V}) \le S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V} \wedge h(\mathrm{V},\Xi)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi))$ whenever $\Xi \le \Theta$.

(2) $\overline{\mathrm{N}} = h$ iff the operator h satisfies the axiom $(RUNGU)$:

$S(\mathrm{U},\mathrm{V}) \le S(h(\mathrm{U},\Theta) \vee h(\mathrm{U},\Xi) \vee \mathrm{U}, h(\mathrm{V},\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes S(h(\mathrm{C}\oplus \mathrm{D},\Theta\odot\Xi), h(\mathrm{C},\Theta) \oplus h(\mathrm{D},\Xi))$ whenever $\Xi \le \Theta$.

Proof. (1) Assume that $\mathrm{N}: \mathrm{X} \to \mathrm{L}^{\mathrm{L}^{\mathrm{X}}\times \mathrm{M}}$ is a reflexive, unary and non-increasing LM-fgns operator on X and $\underline{\mathrm{N}} = h$. Then it follows from the items (RE), (UN) and (NI) of **Theorem 3,** $h(\mathrm{V},\Theta) \le \mathrm{V}$, $h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi) \le h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)$ and $h(\mathrm{V},\Theta) \le h(\mathrm{V},\Xi)$ for each $\mathrm{U},\mathrm{V},\mathrm{C},D \in \mathrm{L}^{\mathrm{X}}$ and $\Theta,\Xi \in \mathrm{M}$ whenever $\Xi \le \Theta$. Also, by **Theorem 4**, we have

$$\begin{aligned} S(\mathrm{U},\mathrm{V}) &\le S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V} \wedge h(\mathrm{V},\Xi)) \otimes h(\underline{\top},\Theta)(x) \otimes \top \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V} \wedge h(\mathrm{V},\Xi)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)). \end{aligned}$$

Thus, h satisfies $(RUNGL)$.

conversely, assume that the operator $h: \mathrm{L}^{\mathrm{X}} \times \mathrm{M} \to \mathrm{L}^{\mathrm{X}}$ is satisfying the axiom $(RUNGL)$. For each $\Theta,\Xi \in \mathrm{M}$ with $\Xi \le \Theta$, if we put $\mathrm{U} = \mathrm{V}$, then we obtain that

$$\top \le S\big(h(\mathrm{V},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V} \wedge h(\mathrm{V},\Xi)\big) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)).$$

This means that

$$\begin{aligned} &\top \le S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)) \\ &\Leftrightarrow \top = S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)) \\ &\Leftrightarrow h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi) \le h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi). \end{aligned}$$

Then it follows from $(RUNGL)$ that

$$\begin{aligned} S(\mathrm{U},\mathrm{V}) &\le S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V} \wedge h(\mathrm{V},\Xi)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)) \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V} \wedge h(\mathrm{V},\Xi)) \otimes h(\underline{\top},\Theta)(x). \end{aligned}$$

Then by **Theorem 10**, there exists a reflexive and non-increasing LM-fgns operator on X such that $\underline{\mathrm{N}} = h$. Further, we have $\forall\ \mathrm{C},\mathrm{D} \in \mathrm{L}^{\mathrm{X}}$ *and* $\Theta,\Xi \in \mathrm{M}$,

$$\begin{aligned} \underline{\mathrm{N}}(\mathrm{C},\Theta) \otimes \underline{\mathrm{N}}(\mathrm{D},\Xi) &= h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi) \\ &\le h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi) = \underline{\mathrm{N}}(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi), \end{aligned}$$

which means that N is unary.

(2)This is similar to proving (1) by utilizing dual properties.

Theorem 19. Let $h: \mathrm{L}^{\mathrm{X}} \times \mathrm{M} \to \mathrm{L}^{\mathrm{X}}$ be an operator, then there exists a reflexive, unary and transitive LM-fgns operator on X such that: for each $\mathrm{U},\mathrm{V},\mathrm{C},\mathrm{D} \in \mathrm{L}^{\mathrm{X}}$, and $\Theta,\Xi \in \mathrm{M}$.

(1) $\underline{\mathrm{N}} = h$ iff the operator h satisfies the axiom $(RUTGL)$:

$$S(\mathrm{U},\mathrm{V}) \le S\big(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V} \wedge h(h(\mathrm{V},\Theta),\Theta)\big) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)).$$

(2) $\overline{\mathrm{N}} = h$ iff the operator h satisfies the axiom $(RUTGU)$:

$$S(\mathrm{U},\mathrm{V}) \le S(h(\mathrm{U},\Theta) \vee \mathrm{U} \vee h(h(\mathrm{U},\Theta),\Theta), h(\mathrm{V},\Theta)) \otimes \neg h(\underline{\perp},\Theta)(x) \otimes S(h(\mathrm{C}\oplus \mathrm{D},\Theta\odot\Xi), h(\mathrm{C},\Theta) \oplus h(\mathrm{D},\Xi)).$$

Proof. (1) Assume that $\mathrm{N}: \mathrm{X} \to \mathrm{L}^{\mathrm{L}^{\mathrm{X}}\times \mathrm{M}}$ is a reflexive, unary and transitive LM-fgns operator on X and $\underline{\mathrm{N}} = h$. Then it follows from the items (RE), (UN) and (TR) of **Theorem 3** $h(\mathrm{V},\Theta) \le \mathrm{V}$, $h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi) \le h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)$ and $h(\mathrm{V},\Theta) \le h(h(\mathrm{V},\Theta),\Theta)$ for each $\mathrm{U},\mathrm{V},\mathrm{C},\mathrm{D} \in \mathrm{L}^{\mathrm{X}}$ and $\Theta,\Xi \in \mathrm{M}$. Also, by **Theorem 4**, we obtain

$$\begin{aligned} S(\mathrm{U},\mathrm{V}) &\le S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta)) \otimes h(\underline{\top},\Theta)(x) \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V} \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes \underline{\top} \\ &= S(h(\mathrm{U},\Theta), h(\mathrm{V},\Theta) \wedge \mathrm{V} \wedge h(h(\mathrm{V},\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(\mathrm{C},\Theta) \otimes h(\mathrm{D},\Xi), h(\mathrm{C}\otimes \mathrm{D},\Theta\odot\Xi)). \end{aligned}$$

Thus, h satisfies $(RUTGL)$.
conversely, assume that the operator $h: L^X \times M \to L^X$ is satisfying the axiom $(RUTGL)$. For each $\Theta, \Xi \in M$, if we put $U = V$, then we get that

$$\top \leq S(h(V,\Theta), h(V,\Theta) \wedge V \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi)).$$

This means that

$$\top \leq S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi))$$
$$\Leftrightarrow \top = S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi))$$
$$\Leftrightarrow h(C,\Theta) \otimes h(D,\Xi) \leq h(C \otimes D, \Theta \odot \Xi)$$

Then it follows from $(RUTGL)$ that

$$S(U,V) \leq S(h(U,\Theta), h(V,\Theta) \wedge V \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x) \otimes S(h(C,\Theta) \otimes h(D,\Xi), h(C \otimes D, \Theta \odot \Xi))$$
$$= S(h(U,\Theta), h(V,\Theta) \wedge h(h(V,\Theta),\Theta)) \otimes h(\underline{\top},\Theta)(x).$$

Then from **Theorem 15**, there exists a reflexive and transitive LM-fgns operator on X such that $\underline{N} = h$. Moreover, we get

$$\underline{N}(C,\Theta) \otimes \underline{N}(D,\Xi) = h(C,\Theta) \otimes h(D,\Xi \leq h(C \otimes D, \Theta \odot \Xi) = \underline{N}(C \otimes D, \Theta \odot \Xi), \ \forall\ C, D \in L^X and\ \Theta, \Xi \in M,$$

which means that N is unary.
(2) This is similar to proving (1) by utilizing dual properties.
Now, we present some results based on **Theorem 19**.
Using the definition of the LM-quasi-fuzzy interior (LM-QFInt) operator on X **[47]**, we find the operator $\underline{N}: L^X \times M \to L^X$ is an LM-QFInt operator iff N is unary, transitive, reflexive, and a non-increasing LM-fgns **[29]**. Therefore, the following result is obtained.

Theorem 20. Let $I: L^X \times M \to L^X$ be an LM-QFInt operator. Then there exists a reflexive, unary and transitive LM-fgns operator $N: X \to L^{L^X \times M}$ such that $\underline{N} = I$ if and only if I is provided with the following property: for each $U, V \in L^X$ and $\Theta \in M$.

$$(INGL)\ S(U,V) \leq S(I(U,\Theta), I(V,\Theta)) \otimes I(\underline{\top},\Theta).$$

Proof. It is clear by **Theorem 1**.
Conversely, from the definition of LM-QFInt operator on X, we get $I(U,\Theta) \leq U$, and $I(U,\Theta) \leq I(I(U,\Theta),\Theta)$ for each $U \in L^X$ and $\Theta \in M$. Because I satisfy $(INGL)$, we find I satisfies $(RUTGL)$. Moreover, from **Theorem 19**, there exists a reflexive, transitive, and unary LM-fgns operator N on X such that $\underline{N} = I$.

By utilizing **Theorem 20** and the relationship between LM - QFInt operator and LM-quasi-fuzzy topology on X **[47]**, we establish the relationship between LM-Rapprox operators based on LM-fgns and LM-quasi-fuzzy topology.
Similarly, using the definition of LM-fuzzy closure (LM-FCl) **[29]** on X, we obtain the operator $\overline{N}: L^X \times M_\perp \to L^X$ be an LM-FCl operator iff N is a non-increasing, reflexive, transitive and unary LM-fgns **[29]**. Therefore, we get the following result.
Theorem 21. Presume $C: L^X \times M_\perp \to L^X$ an LM-FCl operator. Then there exists a reflexive, unary and transitive LM-fgns operator $N: X \to L^{L^X \times M}$ such that $\overline{N} = C$ if and only if C is provided with the following property: for each $U, V \in L^X$, and $\Theta \in M$, where L satisfy the law of double negation.
$(COGL) S(U,V) \leq S(C(U,\Theta), C(V,\Theta)) \otimes \neg C(\underline{\perp},\Theta).$
Proof. It follows immediately from **Theorem 20**.

The same way, by utilizing **Theorem 21** and the relationship between LM-FCl operator and LM-fuzzy co-topology on X **[47]**, we establish the relationship between LM-Rapprox operators based on LM-fgns and LM-fuzzy co-topology.
Remark 3. The results presented in this paper generalize the those introduced by Zhao et al. **[26]**.

4. Conclusions

In this article, we have successfully established single axioms for the characterization of LM-Rapprox operators, generated by non-increasing, unary, reflexive, serial, and transitive LM-fgns. Furthermore, we have presented the connections between LM-Rapprox operators based on LM-fgns and LM-quasi-fuzzy topologies. Our results show that the lower and upper LM-Rapprox operators derived from non-increasing, unary, reflexive, serial, and transitive LM-fgns correspond exactly to a pair of LM-QFInt and LM-FCl operators, respectively.

The findings of this study offer substantial advancements in the theoretical foundations of rough set theory, establishing a solid basis for subsequent investigations and potential real-world applications in contexts characterized by imprecision and indeterminacy. Moreover, the outcomes enrich the fields of artificial intelligence and decision-making. As a direction for future work, these results can be extended to integrate with emerging computational paradigms, such as neutrosophic models, deep learning frameworks, and intelligent decision-support systems.

Competing interests

The author declares that he has no conflicts of interest.

Data Availability

No data were used to support the results of this study.

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