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Analytical study of the nonlinear vibration of an electrostatically actuated microbeam using the extended Galerkin method

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Abstract: Nano/microelectromechanical systems (N/MEMS) have garnered significant attention in recent decades due to their miniature size, potential for batch fabrication, high reliability, and low power consumption. This study examines the nonlinear vibration behavior of an electrostatically actuated clamped-clamped microbeam, described by a second-order nonlinear ordinary differential equation that accounts for both mid-plane stretching and electrostatic forces. In contrast to earlier analytical approaches, which primarily relied on perturbation techniques or purely numerical solutions, the present work applies the extended Galerkin method (EGM) to obtain higher-order approximate solutions for the system's nonlinear dynamic response. The key novelty lies in the implementation of EGM for a strongly nonlinear MEMS configuration and its extension to higher-order terms, which enables the accurate characterization of hardening/softening behaviors and resonance shifts. The derived solutions are validated through comparison with numerical simulations based on the Runge-Kutta method and with existing analytical results, showing that EGM delivers more accurate frequency-amplitude predictions while avoiding small-parameter assumptions and reducing computational effort. The findings underscore the potential of EGM as an efficient and precise analytical approach for the design and performance analysis of nonlinear MEMS resonators.

Keywords: Microelectromechanical systems; Numerical method; Non-linear oscillators; Amplitude frequency formulation; Extended Galerkin method

1. Introduction

The study of non-linear differential equations plays a crucial role in engineering, applied mathematics, engineering, physics and related fields. Significant attention has been given to developing analytical approximation methods for nonlinear oscillators that can provide accurate predictions of both the oscillation frequency and the corresponding solutions.

Microelectromechanical systems (MEMS) have gained considerable attention in recent decades due to their wide-ranging applications in sensing, actuation, and signal processing. One of the fundamental components in many MEMS devices is the microbeam, which often serves as a structural and functional element in devices such as resonators, switches, and sensors. When subjected to electrostatic actuation, these microbeams exhibit complex dynamic behavior, especially under large deflections where geometric nonlinearities become significant [1-4].

In the design and analysis of such systems, accurate modeling of nonlinear vibrations is crucial for predicting performance, ensuring stability, and avoiding failure due to phenomena such as pull-in instability. Classical linear theories are inadequate for capturing the essential dynamics of microbeams experiencing large deformations and nonlinear electrostatic forces. Therefore, advanced analytical methods are required to obtain reliable predictions for such nonlinear systems [5].

Micro-electro-mechanical systems (MEMS) have a wide range of applications, including sensors, accelerometers, biomechanics, microswitches, transistors, electronics, consumer as well as in optical, aerospace, and biomedical engineering, highlighting their significant impact across various fields. Among the different actuation methods employed in MEMS, electrostatic actuation is the most commonly used. This type of actuation is typically modelled by an electrostatically actuated microbeam positioned between a pair of fixed electrodes [6, 7].

Given the importance of the (MEMS), recently there are many an-alytical techniques for solving the MEMS problems, such as frequency-amplitude formulation (FAF) [8], homotopy perturbation method (HPM) [9, 10], variational iteration method (VIM) [11], energy balance method (EBM) [12-14], parameter expansion method (PEM) [15, 16], variational approach (VA) [17], homotopy analysis method (HAM) [18], Hamiltonian approach (HA) [19-21], spreading residue harmonic balance method (SRHBM) [22, 23], global residue harmonic balance method (GRHBM) [24, 25], modified harmonic balance method (MHBM) [26-28], Ateb function (AF) [29], non-perturbative approach (NPA) [30], and so on [31-33].

This study focuses on the analytical investigation of the nonlinear vibration behavior of a fully clamped electrostatically actuated microbeam. The analysis is based on the classical Euler-Bernoulli beam theory, incorporating the effects of mid-plane stretching, which introduces a geometric nonlinearity due to axial strain. Additionally, the electrostatic force is modeled as a nonlinear distributed load that depends inversely on the square of the gap between the beam and the substrate.

To solve the resulting highly nonlinear second-order differential equation, the extended Galerkin Method (EGM) is employed. Unlike the standard Galerkin approach, the EGM provides improved accuracy by enhancing the representation of nonlinear terms, making it particularly suitable for strongly nonlinear MEMS problems. The method reduces the governing equation to a set of nonlinear ordinary differential equations, which are then analyzed to study the dynamic response and stability of the system.

This paper aims to obtain analytical approximate solutions to the large-amplitude vibration of electrostatically actuated micro-beams. Furthermore, results obtained using the EGM are compared with other established analytical and semi-analytical methods and numerical solutions using fourth order Runge-Kutta method, highlighting the advantages and accuracy of the present approach.

2. Nonlinear vibration of an electrostatically actuated microbeam

The governing equation for such a system is a nonlinear second-order differential equation derived from classical beam theory, incorporating the effects of axial stretching and distributed electrostatic loading. In Figure 1, we consider a fully clamped microbeam positioned between two fixed electrodes, with length L, thickness h, and width b. Using classical beam theory and accounting for mid-plane stretching effects along with the distributed electrostatic force, the dimensionless form of the micro-beam's equation of motion can be expressed as presented in [12].

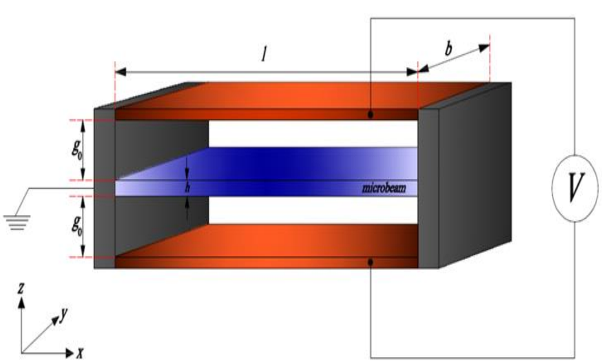


Figure 1: Diagram of a clamped-clamped microbeam-based electromechanical resonator driven from both sides [12].

$$(a_3 + a_2x^2 + a_1x^4)\ddot{x} + a_4x + a_5x^3 + a_6x^5 + a_7x^7 = 0 \quad (1)$$

with

$$x(0) = A,\ \dot{x}(0) = 0, \quad (2)$$

where the coefficients a_1, a_2,…, a_7 can be determined as given in Ref. [12]

$$a_1 = \int_0^1 \phi^6 d\zeta,$$

$$a_2 = -2\int_0^1 \phi^4 d\zeta,$$

$$a_3 = \int_0^1 \phi^2 d\zeta,$$

$$a_4 = \int_0^1 (\phi'''\phi - N\phi''\phi - V^2\phi^2)d\zeta,$$

$$a_5 = -\int_0^1 \left(2\phi''''\phi^3 - 2N\phi''\phi^3 + \alpha\phi''\phi\int_0^1 \phi'^2 d\zeta\right)d\zeta,$$

$$a_6 = -\int_0^1 \left(\phi''''\phi^5 - N\phi''\phi^5 + 2\alpha\phi''\phi^3\int_0^1 \phi'^2 d\zeta\right)d\zeta,$$

$$a_7 = -\int_0^1 \left(\alpha\phi''\phi^5\int_0^1 \phi'^2 d\zeta\right)d\zeta, \quad (3)$$

where the following dimensionless variables and parameters are defined according to Fu et al. [12]:

$$a = \frac{6g_0^2}{h^2},\quad z = \frac{x}{l},\quad N = \frac{\bar{N}l^2}{\bar{E}I},\quad V^2 = \frac{24e_0 l^4\bar{V}^2}{\bar{E}h^3g_0^3} \quad (4)$$

3. Approximate solutions with the extended Galerkin method

It is well known that numerous approximate methods and techniques exist for solving non-linear differential equation (1), and several of these are particularly effective in addressing nonlinear vibration problems. For the problem shown in (1), there are no known efforts even with some popular approximate methods due to the unusual complication of the equation. To test the effectiveness and applicability of the extended Galerkin method (EGM), a standard procedure with the combination of the linear solutions are utilized for the asymptotic solutions as shown in recent studies [34-37].

$$x(t) = \sum_{k=0}^{n} A_{2k+1}\cos(2k+1)\,\omega t, \quad (5)$$

where A_{2k+1}, ω and t are amplitudes, frequency, and time, respectively.

By substituting (5) into (1), the Lagrangian functional is reformulated, allowing the application of the extended Galerkin method (EGM) as in the following formula.

$$\int_0^{2\pi/\omega}\Big(-\Big[a_3 + a_2\Big(\sum_{k=0}^{n} A_{2k+1}\cos(2k+1)\,\omega t\Big)^2 + a_1\Big(\sum_{k=0}^{n} A_{2k+1}\cos(2k+1)\,\omega t\Big)^4\Big]\sum_{k=0}^{n}(2k+1)^2A_{2k+1}\cos(2k+1)\,\omega t + a_4\Big(\sum_{k=0}^{n} A_{2k+1}\cos(2k+1)\,\omega t\Big) + a_5\Big(\sum_{k=0}^{n} A_{2k+1}\cos(2k+1)\,\omega t\Big)^3 + a_6\Big(\sum_{k=0}^{n} A_{2k+1}\cos(2k+1)\,\omega t\Big)^5 + a_7\Big(\sum_{k=0}^{n} A_{2k+1}\cos(2k+1)\,\omega t\Big)^7\Big)\cos(2k+1)\,\omega t\,dt = 0. \quad (6)$$

For application to nano/microelectromechanical systems (N/MEMS), (6) is expressed in the standard form as:

$$\int_0^{T_n}\big((a_3 + a_2x^2 + a_1x^4)\ddot{x} + a_4x + a_5x^3 + a_6x^5 + a_7x^7\big)\cos(2k-1)\omega t\,dt = 0. \quad (7)$$

The solution can be obtained up to higher-order expansions. Through such a procedure, the increase in accuracy can be seen, and solution techniques related to amplitudes can be tested.

4. Solution procedure with the extended Galerkin method

4.1. First order extended Galerkin method

Assume that the first-order approximation of (5) is:

$$x(t) = A_1 \cos \omega t. \qquad (8)$$

Substituting (8) into (2) will result in the amplitude solution as

$$A_1 = A. \qquad (9)$$

Inserting (8) and (9) into (7) with the period of vibration of the first-order mode as $T = \frac{2\pi}{\omega}$,

$$\int_0^{2\pi/\omega} \left((a_3 + a_2 x^2 + a_1 x^4)\, \ddot{x} + a_4 x + a_5 x^3 + a_6 x^5 + a_7 x^7 \right) \cos(\omega t)\, dt = 0. \qquad (10)$$

Resulting in

$$-\frac{5}{8} A^5 \pi \omega a_1 - \frac{3}{4} A^3 \pi \omega a_2 - A \pi \omega a_3 + \frac{A \pi a_4}{\omega} + \frac{3A^3 \pi a_5}{4\omega} + \frac{5A^5 \pi a_6}{8\omega} + \frac{35 A^7 \pi a_7}{64\omega} = 0. \qquad (11)$$

Solving (11), the approximate solution for ω given by

$$\omega = \sqrt{\frac{64a_4 + 48A^2 a_5 + 40A^4 a_6 + 35A^6 a_7}{40A^4 a_1 + 48A^2 a_2 + 64 a_3}}. \qquad (12)$$

As seen in (12), the first-order approximate solution of N/MEMS coincides with that obtained by several existing methods. The solution is valid for small deformations of x(t) and is widely applicable. This confirms that the extended Galerkin method reproduces the same results as other approximate techniques when applied to standard nano/microelectromechanical systems (N/MEMS).

4.2. Second order extended Galerkin method

The second-order approximation for the solution of (7) is taken as follows:

$$x(t) = A_1 \cos \omega t + A_3 \cos 3\omega t, \qquad (13)$$

where A_1 and A_3 are the amplitude. With the boundary condition in (2), it is clear that

$$A_1 = A - A_3. \qquad (14)$$

Now, we extended (7) into two equations as

$$\int_0^{2\pi/\omega} \left((a_3 + a_2 x^2 + a_1 x^4)\, \ddot{x} + a_4 x + a_5 x^3 + a_6 x^5 + a_7 x^7 \right) \cos(\omega t)\, dt = 0, \qquad (15)$$

$$\int_0^{2\pi/\omega} \left((a_3 + a_2 x^2 + a_1 x^4)\, \ddot{x} + a_4 x + a_5 x^3 + a_6 x^5 + a_7 x^7 \right) \cos(3\omega t)\, dt = 0. \qquad (16)$$

Substituting (14) into (15), it yields

$$\omega = \frac{\sqrt{\Delta_1 + \Delta_2}}{2\sqrt{\Delta_3}}, \qquad (17)$$

where

$$\Delta_1 = 64a_4 + 40A^4 a_6 + 35A^6 a_7 - 60A^3 a_6 A_3 - 63A^5 a_7 A_3 + 180A^2 a_6 A_3^2 + 231A^4 a_7 A_3^2 - 220 A a_6 A_3^3,$$

$$\Delta_2 = -469A^3 a_7 A_3^3 + 180 a_6 A_3^4 + 756A^2 a_7 A_3^4 - 714 A a_7 A_3^5 + 364 a_7 A_3^6 + 48 a_5 (A^2 - A A_3 + 2A_3^2),$$

$$\Delta_3 = (a_1(10A^4 + 25A^3 A_3 + 117A^2 A_3^2 - 175 A A_3^3 + 245 A_3^4) + 4(4a_3 + a_2(3A^2 + 5AA_3 + 30A_3^2))).$$

Substituting (13), (14) and (17) into (16) and making necessary simplifications, then the approximate solution of the second-order amplitude is

$$A_3 = (-320A^5 a_1 a_4 - 256A^3 a_2 a_4 - 80A^7 a_1 a_5 + 256A^3 a_3 a_5 + 80A^7 a_2 a_6 + 320A^5 a_3 a_6 + 35A^{11} a_1 a_7 + 112A^9 a_2 a_7 + 336A^7 a_3 a_7)/(2752A^4 a_1 a_4 + 4096A^2 a_2 a_4 + 8192 a_3 a_4 + 1424A^6 a_1 a_5 + 2560A^4 a_2 a_5 + 6144A^2 a_3 a_5 + 1120A^8 a_1 a_6 + 2160A^6 a_2 a_6 + 5440A^4 a_3 a_6 + 1015A^{10} a_1 a_7 + 1988A^8 a_2 a_7 + 5040A^6 a_3 a_7). \qquad (18)$$

It is clear that a different solution in comparisons with earlier studies is obtained in (17), and the approximate one. As it can be seen, the solutions are obtained by the linearization of equations for amplitudes as a simplification.

Further improvement of the solution procedure and results can be done with an iterative procedure, as it has been demonstrated in earlier studies with the extended Galerkin method [34], implying the solutions in (17) can be further improved.

5. Results and Discussion

To assess the accuracy of the extended Galerkin method, we compare the analytical approximate solutions with the numerical fourth order Runge-Kutta method in Figure 2. Additionally, the calculated outcomes are presented and contrasted with existing results from the literature in Table 1.

Table 1 demonstrates the accuracy of the (EGM), highlighting its strong agreement with both the exact analytical solution and other well-known analytical methods. The results presented in Table 1 and Figure 1, confirm that EGM provides highly precise solutions, comparable to those of the homotopy analysis method, while maintaining simplicity and avoiding numerical complexity. Additionally, the differential transformation method is shown to be more accurate than both the variational approach (VA) and the energy balance method (EBM), whose results are found to be less reliable.

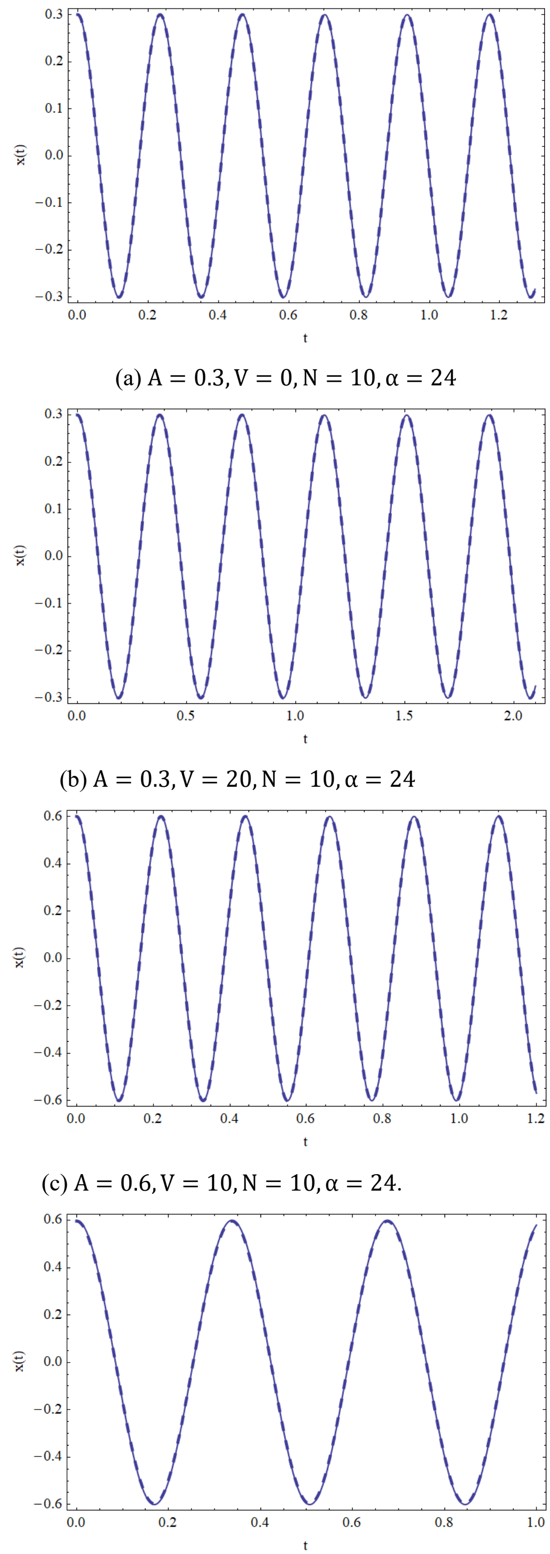
To further understand the influence of amplitude on the conventional MEMS model, the effects of other parameters namely N, V, and α on the second-order approximations obtained using the Extended Galerkin Method (EGM) are examined in more detail. In Figure (3-a), the parameters are set as N=4, 8, 16, 24 with V=5 and α=20. Figure (3-b) presents results for varying V=0, 4, 8, 12 while keeping N=8 and α=20. Figure (3-c) shows the numerical behavior of solutions for different values of α=5, 10, 15, 20 with N=10 and V=20. It is observed that the sensitivity of both first- and second-order approximations to these parameters follows a similar pattern. Therefore, based on the numerical results shown in these figures, it can be concluded that EGM demonstrates stable and consistent performance across various parameter settings.

Figure 4 illustrates the effects of the parameters N, V, and α on the nonlinear frequency of the micro-beam. It is evident that the nonlinear frequency decreases with an increase in the applied voltage (V), while it increases with higher values of axial compressive force (N) and the initial gap (α) between the micro-beam and the fixed electrodes. However, at high values of the applied voltage V, the micro-beam becomes unstable.

Table 1: Comparison of the approximate frequencies with the exact frequencies for different parameter values in (1).

A	N	α	V	ω_{HA} (Error %)	ω_{EBM} (Error %)	ω_{VA} (Error %)	ω_{HAM} (Error %)	ω_{GRHBM} (Error %)	ω_{EGM} (Error %)	ω_{Exact}
Constant Parameters				[21]	[12]	[17]	[18]	[24]	Present	[18]
0.3	10	24	0	26.3669 (1.7837)	26.3867 (1.7073)	26.3644 (1.7933)	26.8372 (0.0000)	26.8372 (0.0000)	26.8372 (0.0000)	26.8372
0.3	10	24	20	16.3547 (1.7970)	16.3829 (1.6218)	16.3556 (1.7914)	16.6486 (0.0000)	16.6486 (0.0000)	16.6486 (0.0000)	16.6486
0.6	10	24	10	26.3562 (8.2789)	26.5324 (7.5598)	26.1671 (9.0614)	28.5368 (0.0049)	28.5378 (0.0014)	28.5391 (0.0050)	28.5382
0.6	10	24	20	17.3013 (7.4497)	17.5017 (6.2194)	17.0940 (8.7528)	18.5902 (0.0000)	18.5902 (0.0000)	18.5962 (0.0322)	18.5902

The analytical expression derived in (13) allows for examining the influence of the parameters defined in (4) on the nonlinear frequency. Figure (4-a) illustrates the effect of the axial load parameter N on the nonlinear frequency as a function of amplitude, with α=24 and V=10. Notably, the impact of N on the frequency is amplitude-dependent. At low amplitudes, increasing the axial tensile load results in a higher nonlinear frequency. However, as the amplitude approaches unity, the frequency becomes nearly independent of N. Furthermore, for each value of axial load, there exists specific amplitude at which the frequency reaches its maximum. Figure (4-b) presents the influence of the electrostatic load parameter V on the nonlinear frequency, with α=24 and N=10. It shows that increasing V leads to a decrease in frequency at given amplitude. Interestingly, for each value of V, the frequency first increases with amplitude, then decreases, indicating a peak frequency at particular amplitude for each electrostatic load. Figure (4-c) shows the effect of the parameter α on the nonlinear frequency, with N=V=10. It is observed that at small amplitudes, the frequency remains unaffected by changes in α. Since α represents the ratio of the initial gap (g_0) to the beam thickness h, this suggests that variations in the initial gap do not significantly influence the frequency at low amplitudes for a given beam geometry. A similar trend is observed at amplitudes near unity. In all cases, the maximum frequency occurs around the amplitude of 0.8 for each value of α.



(a) $A = 0.3, V = 0, N = 10, \alpha = 24$

(b) $A = 0.3, V = 20, N = 10, \alpha = 24$

(c) $A = 0.6, V = 10, N = 10, \alpha = 24$.

(d) $A = 0.6, V = 20, N = 10, \alpha = 24$.

Figure 2: Comparison of analytical approximate solutions using the EGM (dashed line) with numerical solutions (solid line).

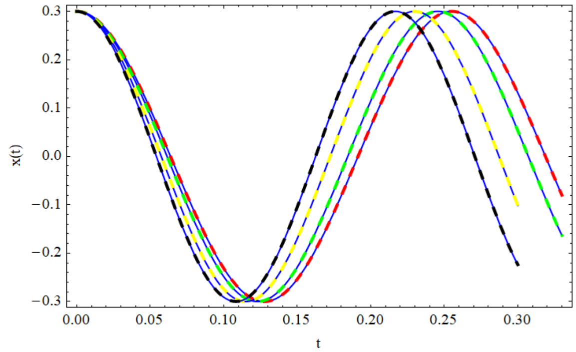


Figure (3-a): Effect of parameter N on the nonlinear frequency at $V = 5, \alpha = 20$.

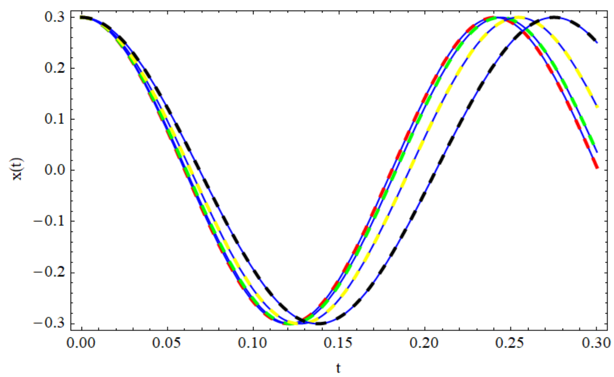


Figure (3-b): Effect of parameter V on the nonlinear frequency at $N = 8, \alpha = 20$.

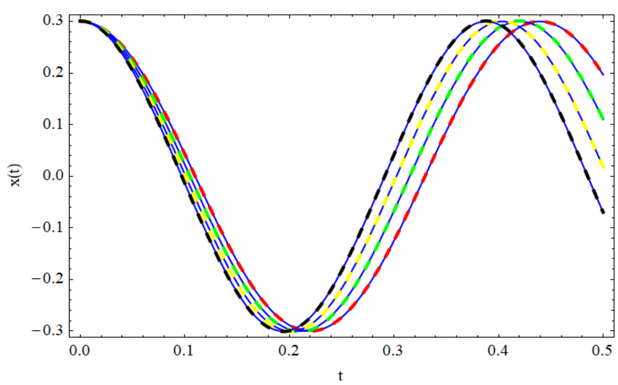
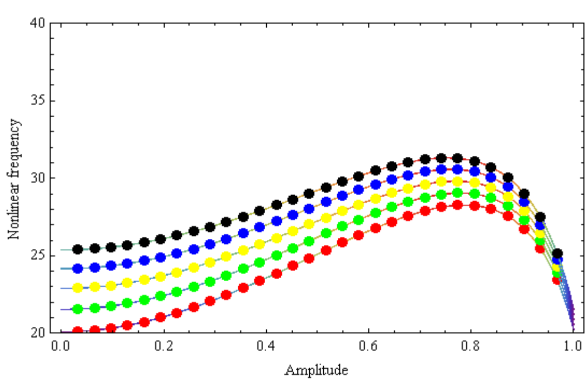
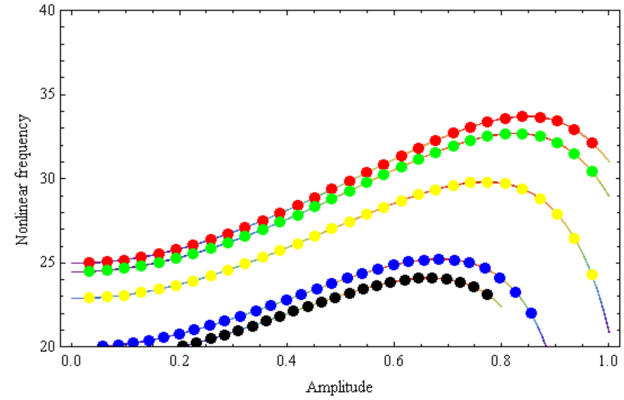


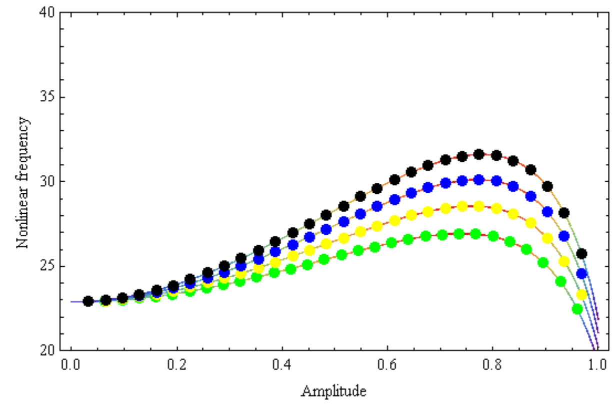
Figure (3-c): Effect of parameter α on the nonlinear frequency at $N = 10, V = 20$.



(a) $V = 10, \alpha = 24$.



(b) $N = 10, \alpha = 24$.



(c) $N = 10, V = 10$.

Figure 4: Effects of the parameters N, V, and α on the nonlinear frequency:

6. Conclusion

This study employs the extended Galerkin method (EGM) to analyze the nonlinear vibration of an electrostatically actuated clamped-clamped microbeam, incorporating geometric nonlinearity from mid-plane stretching and nonlinear electrostatic forces. Higher-order approximate solutions are derived and benchmarked against numerical simulations and other analytical techniques, showing excellent agreement and enhanced accuracy in predicting frequency-amplitude characteristics.

The novelty of this work lies in applying EGM to a strongly nonlinear MEMS configuration and extending it to higher-order terms, enabling precise characterization of hardening/softening effects and resonance shifts without relying on small-parameter assumptions. Compared with existing approaches, the proposed method offers a unique combination of analytical simplicity, high accuracy, and reduced computational cost while preserving physical realism. The results demonstrate that EGM is a robust, efficient, and accurate analytical framework for modeling and optimizing nonlinear MEMS resonators, providing valuable guidance for both theoretical studies and practical device engineering.

Conflict of interests

The author declares that he has no conflicts of interest.

Data Availability

No data were used to support the results of this study.

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