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The k-Confluent Hypergeometric Function and its properties in Bicomplex Numbers

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Abstract:

In this paper, we examine a specialized form of the bicomplex hypergeometric function, known as the k-bicomplex confluent hypergeometric function (CHF). We introduce a detailed analysis of its properties, focusing on its formulation with bicomplex parameters, convergence conditions, and derivative and integral representations. By exploring the k-confluent case, we highlight unique theoretical insights and practical applications, particularly within the framework of bicomplex k-Riemann-Liouville (R-L) Fractional calculus. Our findings expand the current understanding of bicomplex functions in applied sciences and mathematical analysis, laying a foundation for further exploration in specialized functions and fractional operators within the bicomplex domain.

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1 Introduction

The exploration of bicomplex numbers and fractional calculus has led to substantial advancements in applied sciences and mathematical analysis. Bicomplex numbers, introduced by Segre in 1892 [1], extend the complex number field to a four-dimensional framework, allowing for complex-valued coefficients and multiple imaginary units. This number system has broad applications in areas such as special relativity, fluid dynamics, and electromagnetic theory. In recent years, several studies have established foundational aspects of bicomplex function theory, including the idempotent representation, which simplifies calculations involving bicomplex numbers by enabling term-by-term operations [2,3,4,5,6].

Special functions [7,8] form a core area within bicomplex analysis, and extending classical functions to bicomplex variables has proven to be highly beneficial. For instance, the binomial theorem formula, Gauss multiplication theorem, and Legendre duplication provide essential frameworks for expanding the gamma and beta functions to bicomplex numbers, as discussed in [9]. Such extensions have facilitated the development of a more robust theory of bicomplex special functions, allowing for new applications and deeper theoretical insights. Additionally, generalized functions, including the k-generalized beta and gamma functions and the k-Pochhammer symbol, presented by Wu-Sheng and Rafael in 2007 [10], were recently extended to the bicomplex domain in our previous work [11]. These generalizations have created a versatile framework for the study of special functions, enabling new recurrence relations and identities that further enrich the field.

Research on hypergeometric functions has seen a resurgence, as evidenced by the considerable number of recent publications exploring their applications and generalizations. Hypergeometric functions are vital in mathematical analysis, given their wide-ranging applications and complex relationships with other special functions and fractional calculus. Many researchers have developed extensions of hypergeometric functions and introduced various k-symbols and k-fractional derivatives, significantly broadening the scope of these functions [12,13,14,15].

Additionally, Coloma [16] has contributed to fractional bicomplex calculus by defining the (R-L) derivative along the bicomplex basis. This expansion includes basic functions such as exponentials, trigonometric functions, and analytic polynomials, reflects the growing interest in fractional calculus as a nearly foundational area of mathematics, alongside classical calculus. Historically viewed with skepticism, fractional operators have gained recognition in recent decades, as differential equations incorporating fractional operators have proven highly effective for modeling real-world phenomena. Applications of fractional calculus span numerous fields, encompassing biological and ecological modeling, diffusion, control systems, signal processing, and viscoelastic systems.

Building on these foundations, special functions have emerged as a rich area of research within bicomplex analysis, especially with regard to their generalized forms. Generalized special functions are invaluable because many familiar special functions can be seen as specific cases. In particular, the hypergeometric function has gained renewed interest due to its wide applicability in mathematical analysis and its intricate relationships with fractional calculus. This paper builds upon our previous work [17], where we extended the concept of hypergeometric functions to bicomplex variables and developed integral and differential representations for bicomplex hypergeometric functions.

In this work, we delve into a special case of the bicomplex hypergeometric function, termed the k-bicomplex (CHF). This function inherits many essential features of the hypergeometric function while offering simplifications that broaden its applicability. We analyze its convergence region, derive series expansions, and present integral and differential representations specifically adapted to the k-confluent case within the bicomplex framework. Additionally, we extend the theory by incorporating the k-(R-L) fractional derivative and integral in the bicomplex domain, proving several essential theorems to establish a robust theoretical basis for this function.

The k-bicomplex (CHF) holds promise for applications in fields where differential equations and special functions involving bicomplex numbers are relevant, such as quantum mechanics, electromagnetism, and control theory. By extending the framework of bicomplex analysis, our findings contribute to the foundational understanding of bicomplex numbers and fractional operators, providing a solid platform for further exploration in both pure and applied contexts.

2 Preliminaries

The vocabulary and important definitions utilized to produce the primary findings are introduced in this section.

2.1 Bicomplex numbers

A group of bicomplex numbers $\mathbb{BC}$ that result from Segre's work is defined as (see [1,5,6]):

$$\mathbb{BC} = \{\lambda = a_1 + jb_1, a_1,\ b_1 \in \mathbb{C}\}. \tag{1}$$

where $a_1 = \varepsilon_1 + i\varepsilon_2$, $b_2 = \varepsilon_3 + i\varepsilon_4$ and i, j are independent imaginary units defines as

$$i^2 = -1 = j^2,\ ij = ji = k,\quad ik = -j,\quad jk = -i.$$

Idempotent Representation

As indicated by $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$. The two idempotent zero-divisors elements have the following traits (see [4,5,6]):

$$e_1 \cdot e_2 = 0,\ \ e_2 \cdot e_1 = 0,$$

$$e_1^2 = e_1,\ e_2^2 = e_2,$$

$$e_1 + e_2 = 1,\ e_1 - e_2 = ij.$$

The bicomplex number can therefore be written as

$$\lambda = a_1 + jb_1 = \lambda_1 e_1 + \lambda_2 e_2, \tag{2}$$

where $\lambda_1,\ \lambda_2 \in \mathbb{C}$, as $= a_1 - ib_1,\ \lambda_2 = a_1 + ib_1$.

By converting bicomplex numbers into complex numbers, idempotent representations (2) make computations easier. The identities $e_1 \cdot e_2 = 0$ and $e_2 \cdot e_1 = 0$ allow us to write certain significant idempotent qualities as follows: If $\lambda = \lambda_1 e_1 + \lambda_2 e_2$ and $\delta = \delta_1 e_1 + \delta_2 e_2$, then (see [5,6])

1. $\lambda \otimes \delta = \lambda_1\delta_1\ e_1 + \lambda_2\delta_2\ e_2$,
2. $\lambda + \delta = (\lambda_1 + \delta_1)e_1 + (\lambda_2 + \delta_2)e_2$,
3. $\lambda^n = \lambda_1^n e_1 + \lambda_2^n e_2$,
4. $e^{\lambda} = e^{\lambda_1} e_1 + e^{\lambda_2} e_2$,
5. $\frac{1}{\lambda} = \frac{1}{\lambda_1 e_1 + \lambda_2 e_2} = \frac{1}{\lambda_1} e_1 + \frac{1}{\lambda_2} e_2$.

2.2 Special functions

The gamma and beta functions were previously introduced, respectively, as follows (see [19,20]):

$$\Gamma(u) = \int_0^{\infty} e^{-p} p^{u-1}\, dp. \tag{3}$$

This holds true for $u > 0$.

$$\beta(u,v) = \int_0^1 p^{u-1}(1-p)^{v-1}\, dp. \tag{4}$$

It holds true for both $u > 0$ and $v > 0$.

The k-gamma and k-beta functions were defined by Eddy, Rafael, and Wu-Sheng a few years later (see [10], [21]), beginning with the k-Pochhammer symbol as:

$$(\delta_1)_{c,k} = \delta_1(\delta_1 + k)(\delta_1 + 2k)\cdots(\delta_1 + (c-1)k), \tag{5}$$

where $(\delta_1)_0 = 1$, $k > 0$, $c \geqslant 1$, and $\delta_1 \in \mathbb{C} - \{0\}$.

The k-gamma function is defined as [10]

$$\Gamma_k(\delta_1) = \int_0^{\infty} e^{-\frac{p^k}{k}} p^{\delta_1 - 1}\, dp, \tag{6}$$

where $k > 0$, $\delta_1 \in \mathbb{C}$, and $\mathrm{Re}(\delta_1) > 0$.

The k-beta function is defined as [10]

$$\beta_k(\delta_1,\delta_2)=\frac{1}{k}\int_0^1 p^{\frac{\delta_1}{k}-1}(1-p)^{\frac{\delta_2}{k}-1}dp, \tag{7}$$

$$\beta_k(\delta_1,\delta_2)=\frac{\Gamma_k(\delta_1)\Gamma_k(\delta_2)}{\Gamma_k(\delta_1+\delta_2)},\ k>0, \tag{8}$$

where $\delta_1,\delta_2\in\mathbb{C}$ with $\mathrm{Re}(\delta_1)>0$ and $\mathrm{Re}(\delta_2)>0$.

Rainville in [19] introduced the first definition of confluent hypergeometric function as:

$$F(\delta_1;\delta_2;\lambda_1)=\sum_{c=0}^{\infty}\frac{(\delta_1)_c}{(\delta_2)_c}\otimes\frac{\lambda_1^c}{c!}, \tag{9}$$

where $\delta_1,\ \delta_2\ and\ \lambda_1\in\mathbb{C}$.

Gamma and beta functions in bicomplex numbers were defined by Mathur and Goyal in 2006.

The bicomplex gamma function's integral form is explained by (see [9]).

$$\Gamma_2(\lambda)=\int_{\mathbb{D}}e^{-\psi}\otimes\psi^{\lambda-1}\otimes d\psi, \tag{10}$$

where $\lambda=\lambda_1e_1+\lambda_2e_2$, $\psi=\psi_1e_1+\psi_2e_2$, $\psi_1,\psi_2\in\mathbb{R}^+$, and $\mathbb{D}$ be a domain in $\mathbb{BC}$, $\mathbb{D}=(d_1,d_2)$, with $d_1\equiv d_1(\psi_1)$ and $d_2\equiv d_2(\psi_2)$.

For a bicomplex number λ, the Pochhammer symbol is represented by (see [9]):

$$\begin{aligned}(\lambda)_c&=\lambda\otimes(\lambda+1)\otimes(\lambda+2)\otimes(\lambda+3)\otimes\dots\\&\otimes(\lambda+c-1),c\geq 0.\end{aligned} \tag{11}$$

According to [9], The following is the definition of the bicomplex beta function:

$$\beta_2(\lambda,\delta)=\int_{\mathbb{D}}\psi^{\lambda-1}\otimes(1-\psi)^{\delta-1}\otimes d\psi, \tag{12}$$

$$\beta_2(\lambda,\delta)=\frac{\Gamma_2(\lambda)\otimes\Gamma_2(\delta)}{\Gamma_2(\lambda+\delta)}, \tag{13}$$

where $\lambda,\delta,\psi\in\mathbb{BC}$, $\psi=\psi_1e_1+\psi_2e_2$, $\psi_1,\psi_2\in\mathbb{R}^+$, and $\mathbb{D}=(d_1,d_2)$ with $d_1\equiv d_1(\psi_1)$ and $d_2\equiv d_2(\psi_2)$.

Numerous advancements and applications in special functions have occurred recently. In 2024, Bakhet et la. extended the bicomplex gamma and beta functions (see, [11]) as following:

$$\Gamma_{2,k}(\lambda)=\int_{\mathbb{D}}e^{-\frac{\psi^k}{k}}\otimes\psi^{\lambda-1}\otimes d\psi, \tag{14}$$

where $\lambda=\lambda_1e_1+\lambda_2e_2$, $\mathrm{Re}(\lambda_1)>0$ and $\mathrm{Re}(\lambda_2)>0$, $\psi=\psi_1e_1+\psi_2e_2$, $\psi_1,\psi_2\in\mathbb{R}^+$, and $\mathbb{D}=(d_1,d_2)$ with $d_1\equiv d_1(\psi_1)$ and $d_2\equiv d_2(\psi_2)$.

They then provided the following definition of the bicomplex number represented by the k-Pochhammer symbol (see [11]):

$$\begin{aligned}(\lambda)_{c,k}&=\lambda\otimes(\lambda+k)\otimes(\lambda+2k)\otimes(\lambda+3k)\otimes\dots\\&\otimes(\lambda+(c-1)k)\\&=(\lambda_1)_{c,k}\,e_1+(\lambda_2)_{c,k}\,e_2.\end{aligned} \tag{15}$$

The following is the definition of the k-bicomplex beta function:

$$\beta_{2,k}(\lambda,\delta)=\frac{1}{k}\int_{\mathbb{D}}\psi^{\frac{\lambda}{k}-1}\otimes(1-\psi)^{\frac{\delta}{k}-1}\otimes d\psi, \tag{16}$$

where $k\in\mathbb{R}^+$, $\lambda,\delta\in\mathbb{BC}$, $\lambda=\lambda_1e_1+\lambda_2e_2$, $\delta=\delta_1e_1+\delta_2e_2$ with $\mathrm{Re}(\lambda_1)>|\mathrm{Im}(\lambda_2)|$ and $\mathrm{Re}(\delta_1)>|\mathrm{Im}(\delta_2)|$, $\psi=\psi_1e_1+\psi_2e_2$, and $\psi_1,\psi_2\in[0,1]$.

In 2022, Rekha and Ajit defined the bicomplex hypergeometric function (see [18]):

$$F(\zeta,\delta;\eta;\lambda)=\sum_{c=0}^{\infty}\frac{(\zeta)_c(\delta)_c}{(\eta)_c}\otimes\frac{\lambda^c}{c!}, \tag{17}$$

where $\zeta,\delta\ and\ \lambda\in\mathbb{BC}$.

Recently, authors [22] defined bicomplex (CHF) as following

$$F(\zeta;\delta;\lambda)=\sum_{c=0}^{\infty}\frac{(\zeta)_c}{(\delta)_c}\otimes\frac{\lambda^c}{c!}, \tag{18}$$

where $\zeta,\delta\ and\ \lambda\in\mathbb{BC}$; $\delta=\delta_1e_1+\delta_2e_2$, which δ_1,δ_2 are neither zero nor a negative integer.

3 k-Bicomplex Confluent Hypergeometric Function (CHF)

The extension of the confluent function in $\mathbb{BC}$ is covered in this section, along with some fundamental ideas about these functions. We also find the derivative and integral representations of the k-bicomplex (CHF) and find its convergence region with some corollaries.

Theorem 1. *Let ζ, δ, and $\lambda\in\mathbb{BC}$, $\lambda=a_1+j\,b_1=\lambda_1e_1+\lambda_2e_2$, $\delta=a_2+j\,b_2=\delta_1e_1+\delta_2e_2$, $\zeta=a_3+j\,b_3=\xi_1e_1+\xi_2e_2$, then the k-bicomplex (CHF) yields as*

$$\begin{aligned}\mathscr{F}^k(\zeta;\delta;\lambda)&=\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\lambda^c}{c!}\\&=\{F^k(\zeta_1;\delta_1;\lambda_1)\}\,e_1+\{F^k(\zeta_2;\delta_2;\lambda_2)\}\,e_2,\end{aligned} \tag{19}$$

where δ_1, δ_2 are neither zero nor a negative integer, and $a_1,a_2,a_3,b_1,b_2,b_3\in\mathbb{C}$.

Proof. Using the idempotent components associated with e_1 and e_2, the hyperbolic units to duplicate the k-bicomplex (CHF). For bicomplex numbers

$\zeta,\ \delta,\ and\ \lambda$, we get

$$\begin{aligned}\mathscr{F}^k(\zeta;\delta;\lambda) &= \sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\lambda^c}{c!}\\ &= \sum_{c=0}^{\infty}\frac{(\xi_1e_1+\xi_2e_2)_{c,k}}{(\delta_1e_1+\delta_2e_2)_{c,k}}\otimes\frac{(\lambda_1e_1+\lambda_2e_2)}{c!}\\ &= \sum_{c=0}^{\infty}\frac{\left((\xi_1)_{c,k}\,e_1+(\xi_2)_{c,k}\,e_2\right)}{\left((\delta_1)_{c,k}\,e_1+(\delta_2)_{c,k}\,e_2\right)}\\ &\qquad\otimes\frac{(\lambda_1e_1+\lambda_2e_2)}{c!}\\ &= \sum_{c=0}^{\infty}\frac{(\lambda_1(\xi_1)_{c,k})\,e_1+((\lambda_2\ \xi_2)_{c,k})\,e_2}{c!\ (\delta_1)_{c,k}\,e_1+c!\ (\delta_2)_{c,k}\,e_2}\\ &= \left(\sum_{c=0}^{\infty}\frac{(\xi_1)_{c,k}}{(\delta)_{c,k}}\frac{(\lambda_1)}{c!}\right)e_1\\ &\qquad+\left(\sum_{c=0}^{\infty}\frac{(\xi_1)_{c,k}}{(\delta_1)_{c,k}}\frac{(\lambda_1)}{c!}\right)e_2\\ &= \{F^k(\zeta_1;\delta_1;\lambda_1)\}\ e_1+\{F^k(\zeta_2;\delta_2;\lambda_2)\}\ e_2.\end{aligned}$$

Hence, the proof is completed.

The k-bicomplex (CHF) can also be rewritten as

$$\mathscr{F}^k(\zeta;\ \delta;\lambda) = \sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\lambda^c}{c!} = \sum_{c=0}^{\infty}\sigma_{c,k}\otimes\lambda^c,$$

which

$$\sigma_{c,k} = \frac{(\zeta)_{c,k}}{(\delta)_{c,k}\otimes c!} = \sigma_{1,c,k}\ e_1+\sigma_{2,c,k}\ e_2,$$

and

$$\sigma_{1,c,k} = \frac{(\xi_1)_{c,k}}{(\delta_1)_{c,k}c!},\quad \sigma_{2,c,k} = \frac{(\xi_2)_{c,k}}{(\delta_2)_{c,k}c!}.$$

Next, we study about this series convergence region.

Corollary 2. *The series (19) is hyperbolically convergent in the ball $B_N(0,\infty) = \{\lambda : |\lambda|_N <_N \infty\}$ and diverges outside of its closure. Here, $|\lambda|_N$ denotes the modulus of hyperbolic-valued and $|\lambda|_N <_N \infty$ means that λ can take any finite value in the bicomplex space (see [3]).*

Proof. Assume that $\lambda = \lambda_1\ e_1+\lambda_2\ e_2$, $\delta = \delta_1\ e_1+\delta_2\ e_2$, $\zeta = \xi_1e_1+\xi_2e_2$, where $\xi_1,\xi_2,\lambda_1,\lambda_2,\delta_1,\delta_2$ are complex numbers. According to [23], we can use the root and ration test to get

$$R = \lim_{c\to\infty} sup\frac{|\sigma_{c,k}|_N}{|\sigma_{c+1,k}|_N},\ provided\ \delta+r,\zeta+t\notin\mathbb{O}_2.$$

$$R = \left\{\lim_{c\to\infty} sup\left|\frac{\sigma_{1,c,k}}{\sigma_{1,c+1,k}}\right|\right\}e_1+\left\{\lim_{c\to\infty} sup\left|\frac{\sigma_{2,c,k}}{\sigma_{2,c+1,k}}\right|\right\}e_2$$

$$= \left\{\lim_{c\to\infty} sup\left|\frac{(\xi_1)_{c,k}}{(\delta_1)_{c,k}\ c!}\ \frac{(\delta_1)_{c+1,k}\ (c+1)!}{(\xi_1)_{c+1,k}}\right|\right\}e_1$$

$$+\left\{\lim_{c\to\infty} sup\left|\frac{(\xi_2)_{c,k}}{(\delta_2)_{c,k}\ c!}\ \frac{(\delta_2)_{c+1,k}\ (c+1)!}{(\xi_2)_{c+1,k}}\right|\right\}e_2.$$

It follows that $(\xi_1)_{c+1,k} = (\xi_1+ck)(\xi_1)_{c,k}$

$$\begin{aligned}R &= \left\{\lim_{c\to\infty} sup\left|\frac{(\delta_1+ck)(c+1)}{(\xi_1+ck)}\right|\right\}e_1\\ &\qquad+\left\{\lim_{c\to\infty} sup\left|\frac{(\delta_2+ck)(c+1)}{(\xi_2+ck)}\right|\right\}e_2\\ &= \{\infty\}\,e_1+\{\infty\}\,e_2 = \infty.\end{aligned}$$

In the ball $B_N(0,\infty) = \{\lambda : |\lambda|_N <_N \infty\}$, the series is completely hyperbolically convergent and does not diverge anywhere in the bicomplex plane (see [23]). This is the proof.
Which $\mathbb{O}_2$ is the set of Zero divisors defined in [17].

Theorem 3. *Suppose that ζ and δ are bicomplex numbers, then Integral represention of k-bicomplex (CHF) is given by:*

$$\begin{aligned}\mathscr{F}^k(\zeta;\delta;\lambda) &= \frac{\Gamma_{2,k}(\delta)}{k\,\Gamma_{2,k}(\zeta)\ \otimes\ \Gamma_{2,k}(\delta-\zeta)}\int_{\mathbb{D}}(1-\psi)^{\frac{\delta-\zeta}{k}-1}\\ &\qquad\otimes\psi^{\frac{\zeta}{k}-1}\otimes e^{\lambda\otimes\psi}\otimes d\psi.\end{aligned}$$

For any finite value λ in the bicomplex space , which $\psi = \psi_1e_1+\psi_2e_2\in\mathbb{BC}$, ψ_1 and $\psi_2\in[0,1]$, and $\mathbb{D}$ is a curve in $\mathbb{BC}$ made up of two components d_1, d_2 in $\mathbb{C}$.

Proof. From the idempotent representation of the k-bicomplex (CHF) and under the above conditions, we have

$$\begin{aligned}\mathscr{F}^k(\zeta;\delta;\lambda) &= \sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\lambda^c}{c!}\\ &= \left\{\sum_{c=0}^{\infty}\frac{(\xi_1)_{c,k}}{(\delta_1)_{c,k}}\otimes\frac{\lambda_1^c}{c!}\right\}e_1\\ &\qquad+\left\{\sum_{c=0}^{\infty}\frac{(\xi_2)_{c,k}}{(\delta_2)_{c,k}}\otimes\frac{\lambda^c}{c!}\right\}e_2.\end{aligned}$$

From the k-Pochhammer symbol's definition we can used $(\lambda_1)_{c,k} = \dfrac{\Gamma_k(\lambda_1+ck)}{\Gamma_k(\lambda_1)}$, then

$$\mathscr{F}^k(\zeta;\delta;\lambda)=\left\{\frac{\Gamma_k(\delta_1)}{\Gamma_k(\xi_1)}\sum_{c=0}^{\infty}\frac{\Gamma_k(\xi_1+ck)}{\Gamma_k(\delta_1+ck)}\times\frac{\lambda_1^c}{c!}\times\frac{\Gamma_k(\delta_1-\xi_1)}{\Gamma_k(\delta_1-\xi_1)}\right\}e_1$$
$$+\left\{\frac{\Gamma_k(\delta_2)}{\Gamma_k(\xi_2)}\sum_{c=0}^{\infty}\frac{\Gamma_k(\xi_2+ck)}{\Gamma_k(\delta_2+ck)}\times\frac{\lambda_2^c}{c!}\times\frac{\Gamma_k(\delta_2-\xi_2)}{\Gamma_k(\delta_2-\xi_2)}\right\}e_2$$
$$=\left\{\frac{\Gamma_k(\delta_1)}{\Gamma_k(\xi_1)\Gamma_k(\delta_1-\xi_1)}\times\sum_{c=0}^{\infty}\frac{\Gamma_k(\xi_1+ck)\times\Gamma_k(\delta_1-\xi_1)}{\Gamma_k(\delta_1+ck)}\times\frac{\lambda_1^c}{c!}\right\}e_1$$
$$+\left\{\frac{\Gamma_k(\delta_2)}{\Gamma_k(\xi_2)\Gamma_k(\delta_2-\xi_2)}\times\sum_{c=0}^{\infty}\frac{\Gamma_k(\xi_2+ck)\times\Gamma_k(\delta_2-\xi_2)}{\Gamma_k(\delta_2+ck)}\times\frac{\lambda_2^c}{c!}\right\}e_2$$
$$=\left\{\frac{\Gamma_k(\delta_1)}{\Gamma_k(\xi_1)\Gamma_k(\delta_1-\xi_1)}\times\sum_{c=0}^{\infty}\beta_k(\xi_1+ck,\delta_1-\xi_1)\times\frac{\lambda_1^c}{c!}\right\}e_1$$
$$+\left\{\frac{\Gamma_k(\delta_2)}{\Gamma_k(\xi_2)\Gamma_k(\delta_2-\xi_2)}\times\sum_{c=0}^{\infty}\beta_k(\xi_2+ck,\delta_2-\xi_2)\times\frac{\lambda_2^c}{c!}\right\}e_2.$$

By using k-beta function definition in eq. 7, we get

$$\mathscr{F}^k(\zeta;\delta;\lambda)=\left\{\frac{\Gamma_k(\delta_1)}{\Gamma_k(\xi_1)\Gamma_k(\delta_1-\xi_1)}\times\sum_{c=0}^{\infty}\frac{1}{k}\int_0^1(1-\psi_1)^{\frac{\delta_1-\xi_1}{k}-1}\psi_1^{\frac{\xi_1+ck}{k}-1}d\psi_1\times\frac{\lambda_1^c}{c!}\right\}e_1$$
$$+\left\{\frac{\Gamma_k(\delta_2)}{\Gamma_k(\xi_2)\Gamma_k(\delta_2-\xi_2)}\times\sum_{c=0}^{\infty}\frac{1}{k}\int_0^1(1-\psi_2)^{\frac{\delta_2-\xi_2}{k}-1}\psi_2^{\frac{\xi_2+ck}{k}-1}d\psi_2\times\frac{\lambda_2^c}{c!}\right\}e_2$$
$$=\left\{\frac{\Gamma_k(\delta_1)}{k\,\Gamma_k(\xi_1)\Gamma_k(\delta_1-\xi_1)}\times\int_0^1(1-\psi_1)^{\frac{\delta_1-\xi_1}{k}-1}\psi_1^{\frac{\xi_1}{k}-1}\times\sum_{c=0}^{\infty}\frac{(\lambda_1\psi_1)^c}{c!}\,d\psi_1\right\}e_1$$
$$+\left\{\frac{\Gamma_k(\delta_2)}{k\,\Gamma_k(\xi_2)\Gamma_k(\delta_2-\xi_2)}\times\int_0^1(1-\psi_2)^{\frac{\delta_2-\xi_2}{k}-1}\psi_2^{\frac{\xi_2}{k}-1}\times\sum_{c=0}^{\infty}\frac{(\lambda_2\psi_2)^c}{c!}\,d\psi_2\right\}e_2$$
$$=\left\{\frac{\Gamma_k(\delta_1)}{k\,\Gamma_k(\xi_1)\Gamma_k(\delta_1-\xi_1)}\times\int_0^1(1-\psi_1)^{\frac{\delta_1-\xi_1}{k}-1}\psi_1^{\frac{\xi_1}{k}-1}e^{\lambda_1\psi_1}\,d\psi_1\right\}e_1$$
$$+\left\{\frac{\Gamma_k(\delta_2)}{k\,\Gamma_k(\xi_2)\Gamma_k(\delta_2-\xi_2)}\times\int_0^1(1-\psi_2)^{\frac{\delta_2-\xi_2}{k}-1}\psi_2^{\frac{\xi_2}{k}-1}e^{\lambda_2\psi_2}\,d\psi_2\right\}e_2$$
$$=\frac{\Gamma_{2,k}(\delta)}{k\,\Gamma_{2,k}(\zeta)\otimes\Gamma_{2,k}(\delta-\zeta)}\otimes\int_{\mathbb{D}}(1-\psi)^{\frac{\delta-\zeta}{k}-1}\otimes\psi^{\frac{\zeta}{k}-1}\otimes e^{\lambda\otimes\psi}\otimes d\psi.$$

This completes the proof.

Remark 4. *If ζ, δ, and $\lambda\in\mathbb{C}$, we obtain the integral representation of k-(CHF) (see, [24]), and if we put $k=1$, we get the integral representation of (CHF) (see, [19]).*

Theorem 5. *Assume that ζ,δ and $\lambda\in\mathbb{BC}$. Then the m^{th} derivative of k-bicomplex (CHF) is given by*

$$\frac{d^m}{d\lambda^m}\left[\mathscr{F}^k(\zeta;\delta;\lambda)\right]=\frac{(\zeta)_{c,k}}{(\lambda)_{c,k}}\otimes\mathscr{F}^k(\zeta+ck;\delta+ck;\lambda),\ k>0. \tag{20}$$

Proof.
When $m=0$, the relation is correct , then when $m=1$, we have

$$\frac{d}{d\lambda}\left[\Psi_k(\zeta;\delta;\lambda)\right]=\frac{d}{d\lambda}\left(\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\lambda^c}{c!}\right)$$
$$=\sum_{c=1}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\lambda^{c-1}}{(c-1)!}\quad(\text{let } c=v+1)$$
$$=\sum_{v=0}^{\infty}\frac{(\zeta)_{v+1,k}}{(\lambda)_{v+1,k}}\otimes\frac{\lambda^v}{(v)!}$$
$$=\frac{\zeta}{\delta}\otimes\sum_{c=0}^{\infty}\frac{(\zeta+k)_{c,k}}{(\lambda+k)_{c,k}}\otimes\frac{\lambda^c}{(c)!},$$

which $(\zeta)_{c+1,k}=\zeta\otimes(\zeta+k)_{c,k}$.

Likewise, we obtain the 2^{nd} differential, which

$$\frac{d^2}{d\lambda^2}\left[\mathscr{F}^k(\zeta;\delta;\lambda)\right]=\frac{d^2}{d\lambda^2}\left(\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\lambda^c}{c!}\right)$$
$$=\frac{\zeta\otimes(\zeta+k)}{\delta\otimes(\delta+k)}\otimes\sum_{c=0}^{\infty}\frac{(\zeta+2k)_{c,k}}{(\delta+2k)_{c,k}}\otimes\frac{\lambda^c}{(c)!}.$$

By differentiation m times, then we obtain the general relation:

$$\frac{d^m}{d\lambda^m}\left[\mathscr{F}^k(\zeta;\delta;\lambda)\right]=\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\sum_{c=0}^{\infty}\frac{(\zeta+ck)_{c,k}}{(\delta+ck)_{c,k}}\otimes\frac{\lambda^c}{c!}$$
$$=\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\mathscr{F}^k(\zeta+ck;\delta+ck;\lambda).$$

This concludes the proof of the theorem.

4 Application

The following is how the authors [17] defined k-Bicomplex R-L Fractional Calculus :

The k-Bicomplex R-L Fractional integration is provided by :

$$ {}_0I^{\alpha}_{k,\lambda}F(\lambda)=\frac{1}{k\,\Gamma_{2,k}(\alpha)}\otimes\int_0^{\lambda}(\lambda-\psi)^{\frac{\alpha}{k}-1}\otimes F(\psi)\otimes d\psi,\ k\in\mathbb{R}^+, \tag{21}$$

where λ,α and $\psi\in\mathbb{BC}$, $\lambda=a_1+jb_1=\lambda_1e_1+\lambda_2e_2$, with $\mathrm{Re}(a_1)>|\mathrm{Im}(b_1)|$, $\alpha=\alpha_1e_1+\alpha_2e_2$, with $\mathrm{Re}(\alpha_1)>0$, $\mathrm{Re}(\alpha_2)>0$. and $\psi=\psi_1e_1+\psi_2e_2$, $\psi_1,\psi_2\in\mathbb{R}^+$.

The fractional derivative k-R-L of a bicomplex function F of order α can be written as follows:

$$({}_0D^{\alpha}_{k,\lambda}F)(\lambda)={}_0D^{\mu}_{k,\lambda}({}_0D^{-(\mu-\alpha)}_{k,\lambda}F(\lambda))=\frac{1}{k\Gamma_{2,k}(\mu-\alpha)}\otimes\frac{d^{\mu}}{d\lambda^{\mu}}\int_0^{\lambda}(\lambda-\psi)^{\frac{\mu-\alpha}{k}-1}\otimes F(\psi)\otimes d\psi, \tag{22}$$

which λ,α and $\psi\in\mathbb{BC}$, $\lambda=a_1+jb_1=\lambda_1e_1+\lambda_2e_2$, with $\mathrm{Re}(a_1)>|\mathrm{Im}(b_1)|$, $\alpha=\sigma_1+j\sigma_2=\alpha_1e_1+\alpha_2e_2$, with $\mathrm{Re}(\alpha_1)>0$, $\mathrm{Re}(\alpha_2)>0$ and $\mu=[\mathrm{Re}(\sigma_1)]+1$. $\psi=\psi_1e_1+\psi_2e_2$, $\psi_1,\psi_2\in\mathbb{R}^+$.

Definition 6. *Assume $\zeta\in\mathbb{BC}$. The bicomplex k-Fox -Wright function is defined by the formula:*

$$ {}_p\Psi^k_p={}_p\Psi^k_q\left[\begin{matrix}(M_n,P_n)_{1,p}\\(N_m,Q_m)_{1,q}\end{matrix};\zeta\right]=\sum_{u=0}^{\infty}\frac{\prod_{n=1}^{p}\Gamma_{2,k}(M_nu+P_n)}{\prod_{m=1}^{q}\Gamma_{2,k}(N_mu+Q_m)}\otimes\frac{\zeta^u}{u!}, \tag{23}$$

where p and q denote the function numerators and denominators, respectively. M_n,N_m,P_n and $Q_m\in\mathbb{BC}, n=1,2,\dots,p; m=1,2,\dots,q$. Such that $1+\sum_{m=1}^{q}Q_m-\sum_{n=1}^{p}P_n\geq 0$.

The bicomplex k-R-L fractional operator is applied to the k-bicomplex (CHF) in this section.

Theorem 7. *Suppose that the bicomplex function $\mathscr{F}^k(\zeta;\delta;\lambda)$ is piecewise continuous on $O'=(0,\infty)$ and integrable on any finite subinterval of $O=[0,\infty)$, then*

$$ {}_0I^{\alpha}_{k,\lambda}[\mathscr{F}^k(\zeta;\delta;\lambda)]=\lambda^{\frac{\alpha}{k}}\otimes\frac{\Gamma_{2,k}(\delta)}{\Gamma_{2,k}(\zeta)}\otimes{}_2\Psi^k_2\left[\begin{matrix}(k,\zeta),(k,k)\\(k,\delta),(k,\alpha+k)\end{matrix};\lambda\right], \tag{24}$$

where $\zeta,\delta,\lambda,\alpha$ and $\psi\in\mathbb{BC}$, $k>0$, $\psi=\psi_1+j\psi_2$, with $\psi_1,\psi_2\in\mathbb{R}^+$.

Proof. Applying the k-R-L fractional operator (21) to the k-bicomplex (CHF) (19), we obtain

$$ {}_0I^{\alpha}_{k,\lambda}[\mathscr{F}^k(\zeta;\delta;\lambda)]={}_0I^{\alpha}_{k,\lambda}\left[\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\lambda^c}{c!}\right]$$
$$=\frac{1}{k\Gamma_{2,k}(\alpha)}\otimes\int_0^{\lambda}(\lambda-\psi)^{\frac{\alpha}{k}-1}\otimes\mathscr{F}^k(\zeta;\delta;\psi)\otimes d\psi$$
$$=\frac{1}{k\Gamma_{2,k}(\alpha)}\otimes\int_0^{\lambda}(\lambda-\psi)^{\frac{\alpha}{k}-1}\otimes\left[\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\psi^c}{c!}\right]\otimes d\psi$$
$$=\frac{1}{k\Gamma_{2,k}(\alpha)}\otimes\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}\,c!}\otimes\int_0^{\lambda}(\lambda-\psi)^{\frac{\alpha}{k}-1}\otimes\psi^c\otimes d\psi$$
$$=\frac{1}{k\Gamma_{2,k}(\alpha)}\otimes\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}\,c!}\otimes\int_0^{\lambda}\left(1-\frac{\psi}{\lambda}\right)^{\frac{\alpha}{k}-1}\otimes\lambda^{\frac{\alpha}{k}-1}\otimes\psi^c\otimes d\psi$$
$$=\left\{\frac{1}{k\Gamma_k(\alpha_1)}\sum_{c=0}^{\infty}\frac{(\xi_1)_{c,k}}{(\delta_1)_{c,k}\,c!}\times\int_0^{\lambda_1}\left(1-\frac{\psi_1}{\lambda_1}\right)^{\frac{\alpha_1}{k}-1}\lambda_1^{\frac{\alpha_1}{k}-1}\psi_1^c\,d\psi_1\right\}e_1$$
$$+\left\{\frac{1}{k\Gamma_k(\alpha_2)}\sum_{c=0}^{\infty}\frac{(\xi_2)_{c,k}}{(\delta_2)_{c,k}\,c!}\times\int_0^{\lambda_2}\left(1-\frac{\psi_2}{\lambda_2}\right)^{\frac{\alpha_2}{k}-1}\lambda_2^{\frac{\alpha_2}{k}-1}\psi_2^c\,d\psi_2\right\}e_2,$$

where $I_1=\int_0^{\lambda_1}\left(1-\frac{\psi_1}{\lambda_1}\right)^{\frac{\alpha_1}{k}-1}\lambda_1^{\frac{\alpha_1}{k}-1}\psi_1^c\,d\psi_1,$

$$I_2=\int_0^{\lambda_2}\left(1-\frac{\psi_2}{\lambda_2}\right)^{\frac{\alpha_2}{k}-1}\lambda_2^{\frac{\alpha_2}{k}-1}\psi_2^c\,d\psi_2.$$

Let $n_1=\frac{\psi_1}{\lambda_1}$, when $\psi_1=0$, $n_1=0$, when $\psi_1=\lambda_1$, $n_1=1$.
Then

$$I_1=\int_0^1(1-n_1)^{\frac{\alpha_1}{k}-1}\lambda_1^{\frac{\alpha_1}{k}+c}n_1^c\,dn_1,$$

similarly $I_2=\int_0^1(1-n_2)^{\frac{\alpha_2}{k}-1}\lambda_2^{\frac{\alpha_2}{k}+c}n_2^c\,dn_2.$

Then

$$
\begin{aligned}
&{}_0I^{\alpha}_{k,\lambda}[\mathscr{F}^k(\zeta;\delta;\lambda)]\\
&=\frac{\lambda^{\frac{\alpha}{k}}}{k\Gamma_{2,k}(\alpha)}\otimes\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\lambda^c}{c!}\otimes\int_0^1(1-N)^{\frac{\alpha}{k}-1}\otimes N^c\otimes dN\\
&=\frac{\lambda^{\frac{\alpha}{k}}}{\Gamma_{2,k}(\alpha)}\otimes\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\lambda^c}{c!}\otimes\boldsymbol{\beta}_{2,k}(\alpha,ck+k)\\
&=\lambda^{\frac{\alpha}{k}}\otimes\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\lambda^c}{c!}\otimes\frac{\Gamma_{2,k}(ck+k)}{\Gamma_{2,k}(\alpha+ck+k)}\\
&=\lambda^{\frac{\alpha}{k}}\otimes\frac{\Gamma_{2,k}(\delta)}{\Gamma_{2,k}(\zeta)}\otimes\sum_{c=0}^{\infty}\frac{\Gamma_{2,k}(\zeta+ck)}{\Gamma_{2,k}(\delta+ck)}\otimes\frac{\Gamma_{2,k}(ck+k)}{\Gamma_{2,k}(\alpha+ck+k)}\otimes\frac{\lambda^c}{c!}\\
&=\lambda^{\frac{\alpha}{k}}\otimes\frac{\Gamma_{2,k}(\delta)}{\Gamma_{2,k}(\zeta)}\otimes{}_2\Psi_2^k\left[\begin{matrix}(k,\zeta),(k,k)\\(k,\delta),(k,\alpha+k)\end{matrix};\lambda\right].
\end{aligned}
$$

Where ${}_2\Psi_2^k$ represent bicomplex k-Fox-Wright function that defined in eq. (23). Thus, the proof is finished.

Theorem 8. *Assume that $\mathscr{F}^k(\zeta;\alpha;\lambda)$ be a piecewise bicomplex function that continuous on $O'=(0,\infty)$, and integrable on any finite subinterval of $O=[0,\infty)$. Consider $\mu=\mathrm{Re}(a_1)+1$, where $\lambda=a_1+jb_1$, $a_1,\ b_1\in\mathbb{C}$, with $\mu-1<\alpha<\mu$, Then*

$$
{}_0D^{\alpha}_{k,\lambda}[\mathscr{F}^k(\zeta;\delta;\lambda)]=\frac{\lambda^{\frac{\mu-\alpha}{k}-\mu}}{k^{\mu}}\otimes\frac{\Gamma_{2,k}(\delta)}{\Gamma_{2,k}(\zeta)}
\otimes{}_3\Psi_3^k\left[\begin{matrix}(k,\zeta),(k,\mu-\alpha+k),(k,k)\\(k,\delta),(k,\mu-\alpha-\mu k+k),(ck,\mu-\alpha+k)\end{matrix};\lambda\right]. \tag{25}
$$

Proof. Applying the bicomplex k-R-L fractional operator (22) to the k-bicomplex (CHF) (19), we obtain

$$
\begin{aligned}
{}_0D^{\alpha}_{k,\lambda}[\mathscr{F}^k(\zeta;\delta;\lambda)]&=\frac{1}{k\Gamma_{2,k}(\mu-\alpha)}\otimes\frac{d^{\mu}}{d\lambda^{\mu}}\\
&\otimes\int_0^{\lambda}(\lambda-\psi)^{\frac{\mu-\alpha}{k}-1}\otimes\mathscr{F}^k(\zeta;\delta;\psi)\otimes d\psi\\
&=\frac{1}{k\Gamma_{2,k}(\mu-\alpha)}\otimes\frac{d^{\mu}}{d\lambda^{\mu}}\\
&\otimes\int_0^{\lambda}(\lambda-\psi)^{\frac{\mu-\alpha}{k}-1}\otimes\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{\psi^c}{c!}\otimes d\psi\\
&=\frac{1}{k\Gamma_{2,k}(\mu-\alpha)}\otimes\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}\,c!}\otimes\frac{d^{\mu}}{d\lambda^{\mu}}\\
&\otimes\int_0^{\lambda}(\lambda-\psi)^{\frac{\mu-\alpha}{k}-1}\otimes\psi^c\otimes d\psi.
\end{aligned}
$$

Let

$$
\begin{aligned}
I&=\int_0^{\lambda}(\lambda-\psi)^{\frac{\mu-\alpha}{k}-1}\otimes\psi^c\otimes d\psi\\
&=\lambda^{\frac{\mu-\alpha}{k}-1}\otimes\int_0^{\lambda}(1-\frac{\psi}{\lambda})^{\frac{\mu-\alpha}{k}-1}\otimes\psi^c\otimes d\psi\\
&=\left\{\lambda_1^{\frac{\mu1-\alpha_1}{k}-1}\int_0^{\lambda_1}(1-\frac{\psi_1}{\lambda_1})^{\frac{\mu1-\alpha_1}{k}-1}\otimes\psi_1^c\otimes d\psi_1\right\}e_1\\
&+\left\{\lambda_2^{\frac{\mu_2-\alpha_2}{k}-1}\int_0^{\lambda_2}(1-\frac{\psi_2}{\lambda_2})^{\frac{\mu_2-\alpha_2}{k}-1}\otimes\psi_2^c\otimes d\psi_2\right\}e_2\\
&=I_1\ e_1+I_2\ e_2,
\end{aligned}
$$

where

$$
I_1=\lambda_1^{\frac{\mu_1-\alpha_1}{k}-1}\int_0^{\lambda_1}(1-\frac{\psi_1}{\lambda_1})^{\frac{\mu_1-\alpha_1}{k}-1}\otimes\psi_1^c\otimes d\psi_1,
$$

$$
I_2=\lambda_2^{\frac{\mu_2-\alpha_2}{k}-1}\int_0^{\lambda_2}(1-\frac{\psi_2}{\lambda_2})^{\frac{\mu_2-\alpha_2}{k}-1}\otimes\psi_2^c\otimes d\psi_2.
$$

Suppose that $n_1=\frac{\psi_1}{\lambda_1}$. When $\psi_1=0,\ n_1=0$ and when $\psi_1=\mu_1,n_1=1$. Then

$$
I_1=\lambda_1^{\frac{\mu_1-\alpha_1}{k}+c}\int_0^1(1-n_1)^{\frac{\mu_1-\alpha_1}{k}-1}\otimes n_1^c\otimes dn_1,
$$

similarly

$$
I_2=\lambda_2^{\frac{\mu_2-\alpha_2}{k}+c}\int_0^1(1-n_2)^{\frac{\mu_2-\alpha_2}{k}-1}\otimes n_2^c\otimes dn_2.
$$

Then

$$
\begin{aligned}
&{}_0D^{\alpha}_{k,\lambda}[\mathscr{F}^k(\zeta;\delta;\lambda)]\\
&=\frac{1}{k\Gamma_{2,k}(\mu-\alpha)}\otimes\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}}\otimes\frac{d^{\mu}}{d\lambda^{\mu}}\left[\lambda^{\frac{\mu-\alpha}{k}+c}\right]\\
&\otimes\int_0^1(1-N)^{\frac{\mu-\alpha}{k}-1}\otimes N^c\otimes dN\\
&=\frac{1}{\Gamma_{2,k}(\mu-\alpha)}\otimes\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}\,c!}\otimes\frac{d^{\mu}}{d\lambda^{\mu}}\left[\lambda^{\frac{\mu-\alpha}{k}+c}\right]\\
&\otimes\boldsymbol{\beta}_{2,k}(\mu-\alpha,ck+k)\\
&=\sum_{c=0}^{\infty}\frac{(\zeta)_{c,k}}{(\delta)_{c,k}\,c!}\otimes\frac{\Gamma_2\left(\frac{\mu-\alpha}{k}+c+1\right)}{\Gamma_2\left(\frac{\mu-\alpha}{k}+c-\mu+1\right)}\\
&\otimes\lambda^{\frac{\mu-\alpha}{k}+c-\mu}\otimes\frac{\Gamma_{2,k}(ck+k)}{\Gamma_{2,k}(\mu-\alpha+ck+k)},
\end{aligned}
$$

using property $\Gamma_{2,k}(\zeta)=k^{\frac{\zeta}{k}-1}\otimes\Gamma_2\left(\frac{\zeta}{k}\right)$, then

$$
\begin{aligned}
{}_0D^{\alpha}_{k,\lambda}[\mathscr{F}^k(\zeta;\delta;\lambda)]&=\lambda^{\frac{\mu-\alpha}{k}-\mu}\sum_{s=0}^{\infty}\frac{(\zeta)_{s,k}}{(\delta)_{s,k}}\\
&\otimes\frac{k^{-\mu}\otimes\Gamma_{2,k}(\mu-\alpha+ck+k)}{\Gamma_{2,k}(\mu-\alpha+ck-\mu k+k)}\\
&\otimes\frac{\Gamma_{2,k}(ck+k)}{\Gamma_{2,k}(\mu-\alpha+ck+k)}\otimes\frac{\lambda^c}{c!}
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{\lambda^{\frac{\mu-\alpha}{k}-\mu}}{k^{\mu}}\otimes\frac{\Gamma_{2,k}(\delta)}{\Gamma_{2,k}(\zeta)}\\
&\otimes{}_3\Psi_3^k\left[\begin{matrix}(k,\zeta),(k,\mu-\alpha+k),(k,k)\\(k,\delta),(k,\mu-\alpha-\mu k+k),(k,\mu-\alpha+k)\end{matrix};\lambda\right].
\end{aligned}
$$

The theorem has been clearly explained.

5 Conclusion

In this paper, we introduced the k-bicomplex (CHF) and developed a foundational theory for this special function within the framework of bicomplex analysis. By applying bicomplex parameters to the (CHF), we established its series representation and convergence properties. We also formulated integral and differential representations specific to the bicomplex k-confluent case, which serve as essential tools for further analysis. Utilizing the k-R-L fractional calculus, we proved several new theorems that extend the utility of fractional operators within the bicomplex domain.

Our results underscore the versatility and potential applications of the k-bicomplex (CHF) in mathematical modeling, physics, and engineering fields where complex systems require advanced analytic tools. By providing a framework for these functions and fractional operators, we contribute to the ongoing expansion of bicomplex special functions, enhancing their applicability in diverse contexts.

Looking forward, further research could explore applications of the k-bicomplex (CHF) in solving differential equations and in mathematical physics, particularly in areas like quantum mechanics, control systems, and electromagnetic theory. Additionally, exploring connections between k-bicomplex (CHF) and other classes of special functions could yield new insights and broaden the theoretical scope of bicomplex analysis. These findings set the stage for the continued development of bicomplex function theory and its applications in both theoretical and applied sciences.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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