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Statistical Inference of a New Pareto-type Model under Generalized Hybrid Type-I Censored Samples

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Abstract:

In life-testing experiments, generalized hybrid Type-I censoring scheme (GHTCS) has been adopted to enhance the statistical efficiency of estimators. The core focus of this article is to extensively tackle the critical matter of estimation the model parameters and the parameters of life (reliability and hazard rate functions) for a new Pareto-type distribution (NPD) based on GHTCS. Initially, the model parameters as well as the parameters of life are estimated by maximum likelihood method and the corresponding approximate confidence intervals are constructed with respected to the observed Fisher information matrix. To address scenarios where sample sizes are small, confidence intervals are created by employing the percentile bootstrap method. In addition, the point and credible intervals estimate of parameters are constructed with respect to symmetric squared error loss Bayes method. To provide a robust and efficient framework for accurate estimation the approximate Bayes estimators are computed under the technique of Markov chain Monte Carlo (MCMC). The efficiency of estimators and comparative analysis of their performance are assessed under constructed the comprehensive simulation study. Ultimately, the application of the estimators is demonstrated through the analysis a set of real data.
keywords: GHTCS, a new Pareto-type distribution, delta method, bootstrap method, Bayesian estimation, MCMC, importance sampling.

1 Introduction

1.1 A new Pareto-type distribution

Pareto, the eminent scholar, revolutionized the realm of economic analysis by introducing the illustrious Pareto distribution as an exemplary model for income distribution Ref. [1]. Distinguished by its remarkable skewness and a profound heavy tail, this distribution remains a pillar in comprehending economic dynamics. Its versatile application has extended to a multitude of domains, encompassing the analysis of extreme environmental events, Changes in insurance claims or financial information, and the evaluation of reliability in diverse contexts. Pareto distribution is also used to predict the lifetime of produced goods with defined guarantee durations. Several books, including those by, Refs. [2–4] have effectively covered the areas of application for the Pareto distribution. Over the past 20 years, numerous investigations on the Pareto distribution have been carried out by different authors, specifically on the estimation and prediction of its parameters. These studies can be seen in articles such as, Refs. [5–10].

In a remarkable advancement, authors of Ref. [11] have introduced a pioneering Pareto-type (NP) distribution that revolutionizes the modeling of income and reliability data. This remarkable development serves as an exceptional generalization of the renowned Pareto distribution, significantly expanding the scope and accuracy of data analysis in diverse fields. Let $\underline{X}$ be a random variable. It has the NP distribution with shape parameter α and scale parameter λ. The following is an expression for $\underline{X}$'s PDF (probability density function);

$$f(x) = 2\alpha\lambda^{\alpha}x^{-(\alpha+1)}(1+(\frac{\lambda}{x})^{\alpha})^{-2}, \quad x \geq \lambda. \tag{1}$$

The random variable $\underline{X}$'s CDF can be shown as;

$$F(x) = 1 - \frac{2(\frac{\lambda}{x})^{\alpha}}{1+(\frac{\lambda}{x})^{\alpha}}. \tag{2}$$

The PDF and CDF enable us to obtain the expressions for the reliability function (RF) and the hazard rate function (HRF) as follows;

$$R(t) = \frac{2\lambda^{\alpha}t^{-\alpha}}{1+(\frac{\lambda}{t})^{\alpha}}, \quad t > 0, \tag{3}$$

and

$$H(t) = \frac{\alpha t^{-1}}{1+(\frac{\lambda}{t})^{\alpha}}, \quad t > 0. \quad (4)$$

The plots in Figs.1 and 2 demonstrate the PDF and HRF of the NP distribution for different values of the shape parameter α, while keeping the scale parameter λ fixed at 1.

As highlighted in [11, 12], the PDF of the NP distribution that decreases for $x \geq \lambda > 0$. If $\alpha > 1$, the hazard rate function (HRF) of a distribution can exhibit a unimodal shape. On the other hand, if $\alpha \leq 1$, the hazard rate function must decrease. These attributes provide a degree of adaptability for researchers to apply the model to practical situations. To delve deeper into the NP distribution and explore recent references on the topic, consider the following sources for more comprehensive information [13–15].

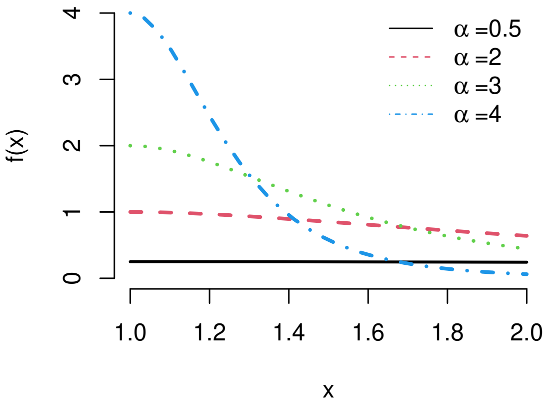


Figure 1. PDF of A new Pareto-type distribution.

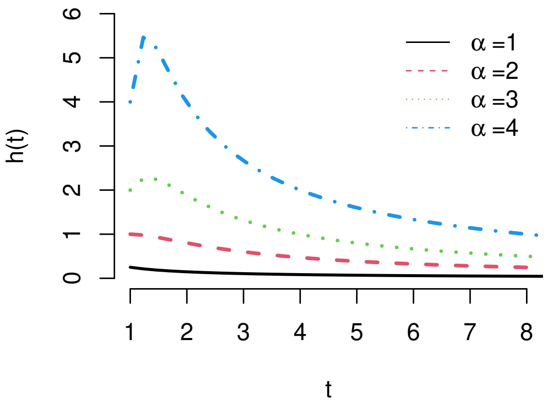


Figure 2. HRF of A new Pareto-type distribution.

1.2 GHT Censoring Scheme

As a result of significant advancements in science and technology, industrial products have achieved exceptional levels of reliability. Nevertheless, the process of acquiring an adequate amount of failure time data for statistical analysis purposes in life testing experiments can be a resource-intensive and time-consuming endeavor. Hence, the need to reduce testing time and costs has led to the use of censoring techniques. A wide range of statisticians have investigated censoring strategies in great detail. Lifetime experiments include ending the experiment at a predetermined period or when a predetermined number of failures have occurred. Type-I and Type-II censoring are widely acknowledged as the two fundamental approaches used in lifetime experiments. As science and technology progresses, products become more dependable and have an extended lifespan, necessitating longer life-testing to gather an adequate number of failure samples. In order to reduce experimental cost and time even further, Ref. [16] employed a hybrid censoring scheme (HCS) that blends the two fundamental censoring schemes discussed previously. Type-I and Type-II hybrid censoring schemes (HCS) are the two categories into which HCSs fall. The use of Type-I and Type-II censoring techniques establishes these groupings. Based on statistical analysis, the HCS can be characterised as follows;

1.Type-I hybrid censored
Let $X_{1:n}, \ldots, X_{n:n}$ represent the ordered lifetimes of the n items in the event that the test consists of several items. The test is concluded and terminated based on two conditions: either when a predetermined number r, of the n items have failed (where $r < n$), or when a pre-determined test time, denoted as τ, has been reached. In other words, the test terminates at a random time $\tau^* =$ min $(X_{r:n}, \tau)$. Furthermore, it's a widely held belief that the items that fail the test are not changed. We have either of the following two kinds of observations:
Case I: If $x_{r:n} \leq \tau$, the observations can be denoted as $\{x_{1:n} < \cdots < x_{r:n}\}$.
Case II: If $x_{r:n} > \tau$, the observations can be denoted as $\{x_{1:n} < \cdots < x_{d:n}\}$.
Let d denote the number of failures noted prior to time point τ.

2.Type-II hybrid censored
Similar to traditional Type-I censoring, the main disadvantage of Type-II hybrid censoring is the inference findings. These results heavily depend on the condition that there is a minimum of one observed failure. Additionally, there's a chance that very few failures will occur during the allotted time. The τ. In such instances, the efficiency of the estimator(s) can be substantially compromised. An alternative hybrid censoring scheme, referred to as Type-II hybrid censored,

has been proposed, which involves terminating the experiment at the random time $T^* = \max\,(X_{r,n}, \tau)$. One of two kinds of observations is in our possession, specifically:

Case I: If $x_{r:n} \geq \tau$, the observations can be denoted as $\{x_{1:n} < \cdots < x_{r:n}\}$.

Case II: If $x_{r:n} < \tau$, the observations can be denoted as $\{x_{1:n} < \cdots < x_{d:n}\}$.

Since $d = n$, in this case, we assume $x_{d+1} = \infty$. Here, $r \leq d \leq n$ indicates the number of detected failures before time τ.

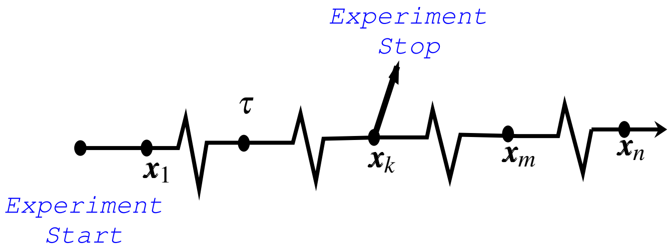
Moreover, these schemes have limitations such as the possibility of only a small number of failures happening before the pre-determined time, or uncertainty about the maximum time required to complete the test. Therefore, it may not be feasible to perform statistical analysis using such a schemes. In order to overcome this limitation and improve the effectiveness of estimators in life-testing experiments, it is necessary to ensure that a predetermined number of failures occur before the experiment concludes. This approach serves to minimize testing time and reduce the associated costs related to unit failures, while simultaneously enhancing the overall efficacy of the estimation process. Ref. [17] introduced a new, efficient censoring scheme called the GHTCS. This scheme guarantees a minimum number of failures and addresses the shortcomings of traditional Hybrid Type-I censoring. Now, let's provide a brief description of GHTCS as following;

Let $(X_1, X_2, \ldots, X_n)$ represent the lifetimes of a set of n units. As independent, identically distributed (i.i.d.) random variables, these lifetimes are considered to be. We denote the ordered lifetime of these units as $(X_1 < X_2 < \cdots < X_n)$. Here, We select two integers, namely k and m, with the condition that $k < m < n$. Additionally, we have a positive time value denoted by τ, which can range from 0 to infinity. In this experiment, our objective is to investigate failure occurrences. If the k^{th} failure happens before the specified time τ, we will conclude the experiment at the minimum value between $X_{(m)}$ (the m^{th} order statistic) and τ. On the other hand, if the k^{th} failure occurs after the time τ, we will terminate the experiment and record the value $X_{(k)}$ (the k^{th} order statistic). This experimental setup ensures that a minimum of k failures will be observed before concluding the experiment. The GHTCS can be reduces into different variations based on values of k, m, τ as follows;

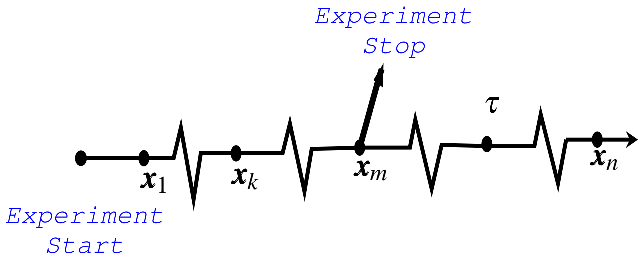
1. When k is allowed to be 0, the GHTCS reduces to the Type-I hybrid censoring scheme.
2. In the case where m equal n, the GHTCS transforms into the modified Type-II hybrid censoring scheme.
3. As the time limit τ approaches infinity, the GHTCS converges to the Type-II censoring scheme.

This article focuses on GHTCS and categorizes it into three scenarios:

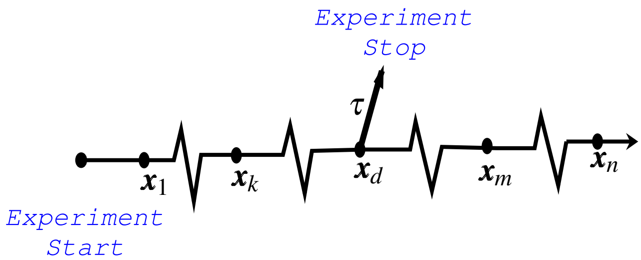
Scenario I: $\{X_1 < \cdots < X_k\}$, with $X_k > \tau$



Scenario II: $\{X_1 < \cdots < X_m\}$, with $X_m < \tau$



Scenario III: $\{X_1 < \cdots < X_d\}$, with $X_k < \tau < x_m$



Accelerated Lomax lifespan distribution with a dependent competing risks model was covered under GHTCS in Refs. [18]. Ref. [19] discussed Generalized Type-I Hybrid Censoring Scheme in Estimation Competing Risks Chen Lifetime Populations. Ref. [20] studied estimations of parameters from Bathtub-Shaped distribution. Furthermore, Ref. [21] applied Constant-Stress Partially Accelerated Life Test Model under Exponentiated Gamma distribution. We consider a life testing experiment involving a randomly selected set of n independent units. In this experiment, we introduce two prior integers, namely k and m, where k is less than m, and m is less than n. The joint likelihood function associated with the GHTCS for the order observed data vector $\underline{X} = (X_1 < X_2 < \cdots < X_D)$ is given by;

$$L(\underline{X}) = \frac{n!((n-D)!)^{-1}}{(1-F(T_D))^{D-n}} \prod_{i=1}^{D} f(x_i) \quad , \tag{5}$$

where

$$(D, T_D) = \begin{cases} (k, X_k) & if \quad \tau < x_k < X_m \\ (m, X_m) & if \quad X_k < x_m < \tau \\ (D^*, T^*) & if \quad x_k < \tau < X_m \end{cases}$$

Let D denote the total number of failures noted in the given context prior to time τ.

This study's main goal is to close the current gap in the literature by accurately estimating the NP distribution's parameters and reliability features. This will be accomplished by analyzing data collected through GHTCS. To achieve accurate results, we will employ three robust methodologies: maximum likelihood, percentile bootstrap, and Bayesian approaches. Our objectives are to compute the hazard rate functions and reliability of the NP distribution using lifetime data gathered under GHTCS, as well as to derive point and interval estimates of unknown parameters. This is the arrangement of the paper: In Section (2), we examine MLEs and asymptomatic confidence intervals ACIs of the unknown parameters. Section (3) applies the bootstrap resampling method to construct two bootstrap confidence intervals. Bayesian estimation method is discussed in section (4). In Section (5), we analyse real censored data and presents numerical simulation results. Finally, the conclusions are summarized in section (6).

2 Classical Inference

2.1 MLE

Finding maximum likelihood estimators for the unknown parameters and estimating the hazard rate function (HRF) and reliability function (RF) under GHTCS are the goals of this subsection. Assume a life test has n units. It is considered that the unit X's life will follow the NP distribution. Under GHTCS, the order failure data of size D, $x_1 < x_2 < \ldots < x_D$ can be obtained. Then, by substituting Eqs. (1) to (4) into Eq. (5), (α,λ)'s corresponding likelihood function can be expressed as follows:

$$L(\alpha,\lambda) \propto \frac{\alpha^D\,\lambda^{\alpha\,n}}{(x_D^\alpha+\lambda^\alpha)^{(n-D)}}\prod_{i=1}^{D}\frac{x_i^{\alpha-1}}{(x_i^\alpha+\lambda^\alpha)^2}, \lambda \le x_{(1)}, \tag{6}$$

where $x_{(1)} = \min\{x_1,\ldots,x_D\}$. Eq.(6) yields the equivalent log-likelihood function $\ell(\cdot)$.

$$\ell(\alpha,\lambda) \propto D\ln\alpha + \alpha n ln\lambda - (n-D)ln(x_D^\alpha+\lambda^\alpha) + (\alpha-1)\sum_{i=1}^{D} ln x_i - 2\sum_{i=1}^{D} ln(x_i^\alpha+\lambda^\alpha) \tag{7}$$

The maximum of the log-likelihood function in Eq. (7) with regard to α and λ is used to calculate the ML estimates of the unknown parameters. The MLE of λ is $\hat{\lambda} = x_{(1)}$ since it is evident that $\ell(\alpha,\lambda)$ increases monotonically with λ. In Eq. (6), we derive the profile log-likelihood function of α and λ without the additive constant as $\hat{\lambda}$.

$$\frac{\partial\ell}{\partial\alpha} = \frac{D}{\alpha} + n\ln\lambda - (n-D)\left[\frac{x_D^\alpha \ln x_D + \lambda^\alpha \ln\lambda}{x_D^\alpha+\lambda^\alpha}\right] + \sum_{i=1}^{D}\ln x_i - 2\sum_{i=1}^{D}\left[\frac{x_i^\alpha\ln x_i + \lambda^\alpha\ln\lambda}{x_i^\alpha+\lambda^\alpha}\right] \tag{8}$$

and

$$\frac{\partial\ell}{\partial\lambda} = \frac{\alpha}{\lambda}\left[(D-n)\left[\frac{\lambda^\alpha}{x_D^\alpha+\lambda^\alpha}\right] + \Lambda(\alpha,\lambda)\right] \tag{9}$$

where

$$\Lambda(\alpha,\lambda) = n - 2\sum_{i=1}^{D}\left[\frac{\lambda^\alpha}{x_i^\alpha+\lambda^\alpha}\right]$$

From Eq. (9), it can be verified that $\Lambda(\alpha,\lambda)$ is non-negative, leading to the conclusion that $\frac{\partial\ell}{\partial\lambda} \ge 0$. This demonstrates that $\ell(\alpha,\lambda)$ increases as λ increases. We may derive the partial log-likelihood function with regard to α by substituting $\hat{\lambda}$ in Eq. (8).

$$\Psi(\alpha) = \frac{\partial\ell}{\partial\alpha} = \frac{D}{\alpha} + n\ln x_{(1)} - 2\sum_{i=1}^{D}\left[\frac{x_i^\alpha\ln x_i + x_{(1)}^\alpha\ln x_{(1)}}{x_i^\alpha + x_{(1)}^\alpha}\right] + \sum_{i=1}^{D}\ln x_i - (n-D)\left[\frac{x_D^\alpha\ln x_D + x_{(1)}^\alpha\ln x_{(1)}}{x_D^\alpha + x_{(1)}^\alpha}\right] \tag{10}$$

Consequently, maximising Eq.(10) yields the MLE of α. In order to compute the estimate $\hat{\alpha}$, various iterative methods can be employed to solve the likelihood equation. An interesting observation is that the maximum likelihood estimator (MLE) α can be obtained as a fixed-point solution of the equation $\Omega(\alpha) = \alpha$, where $\Omega(\alpha)$ represents a specific function.

$$\Omega(\alpha) = \frac{D}{\Xi + 2\sum_{i=1}^{D}\left[\frac{x_i^\alpha\ln x_i + x_{(1)}^\alpha\ln x_{(1)}}{x_i^\alpha + x_{(1)}^\alpha}\right] - n\ln x_{(1)} - \sum_{i=1}^{D}\ln x_i}$$

where, $\Xi = (n-D)\left[\frac{x_D^\alpha\ln x_D + x_{(1)}^\alpha\ln x_{(1)}}{x_D^\alpha + x_{(1)}^\alpha}\right]$

Clearly,

$$\lim_{\alpha\to 0}\Psi(\alpha) = \infty,$$

$$\lim_{\alpha\to\infty}\Psi(\alpha) = (n-d)\ln x_D + \sum_{i=1}^{D}\ln x_i - n\ln x_{(1)} < 0$$

and

$$\Omega^{`}(\alpha) = -\frac{D}{\alpha^2} - (n-D)\,x_D^\alpha\,x_{(1)}^\alpha\left[\frac{(\ln x_D - \ln x_{(1)})^2}{\left(x_D^\alpha + x_{(1)}^\alpha\right)^2}\right] - 2\,x_{(1)}^\alpha\sum_{i=1}^{D}x_i^\alpha\left[\frac{(\ln x_i - \ln x_{(1)})^2}{\left(x_i^\alpha + x_{(1)}^\alpha\right)^2}\right]$$

Consequently, $\Omega(\alpha)$ is a continuous and monotonically decreasing function in the interval $(0,\infty)$. This implies that there exists a unique maximum likelihood estimate for α. After obtaining the estimates $\hat{\alpha}$ and $\hat{\lambda}$ for α and λ, respectively, we can utilize the in-variance property of maximum likelihood estimation. This property allows us to calculate the maximum likelihood estimators $\hat{R}(t)$ and $\hat{H}(t)$ of $R(t)$ and $H(t)$, respectively, as given in Eqs. (3) and (4). These estimators are determined for a specified mission time $t > 0$ by substituting α and λ with their respective maximum likelihood estimates $\hat{\alpha}$ and $\hat{\lambda}$.

$$\hat{R}(t) = \frac{2\hat{\lambda}^{\hat{\alpha}} t^{-\hat{\alpha}}}{1+\left(\frac{\hat{\lambda}}{t}\right)^{\hat{\alpha}}} \quad , \quad \hat{H}(t) = \frac{\hat{\alpha} t^{-1}}{1+\left(\frac{\hat{\lambda}}{t}\right)^{\hat{\alpha}}}$$

2.2 ACIs

Asymptotic normality of the maximum likelihood estimators can be relied upon to find the confidence intervals (CIs) for α and λ with confidence level of $100(1-\gamma)\%$. The inverse of the observed Fisher information matrix can be used to estimate the variances, $\mathrm{Var}(\hat{\alpha})$ and $\mathrm{Var}(\hat{\lambda})$. This matrix is calculated from the second derivatives of the log-likelihood function evaluated at $\hat{\alpha}$ and $\hat{\lambda}$. By employing this approach, we can derive the confidence intervals for α and λ with a desired level of confidence. The second derivatives of ℓ with respect to α and λ are calculated using Eq. (7) as;

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{D}{\alpha^2} - (n-D)\left(\frac{x_D}{x_{(1)}}\right)^{\alpha}\left[\frac{\ln\left(\frac{x_D}{x_{(1)}}\right)}{1+\left(\frac{x_D}{x_{(1)}}\right)^{\alpha}}\right]^2 - 2\sum_{i=1}^{D}\left(\frac{x_i}{x_{(1)}}\right)^{\alpha}\left[\frac{ln\left(\frac{x_i}{x_{(1)}}\right)}{1+\left(\frac{x_i}{x_{(1)}}\right)^{\alpha}}\right]^2, \tag{11}$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \alpha} = \frac{n}{\lambda} + (n-D)\left(\frac{x_D}{\lambda}\right)^{\alpha}\left[\frac{(\ln\lambda)\ \ln\left(\frac{x_D}{\lambda}\right)}{\left(1+\left(\frac{x_D}{\lambda}\right)^{\alpha}\right)^2}\right] + 2\sum_{i=1}^{D}\left(\frac{x_i}{\lambda}\right)^{\alpha}\left[\frac{(\ln\lambda)\ \ln\left(\frac{x_i}{\lambda}\right)}{\left(1+\left(\frac{x_i}{\lambda}\right)^{\alpha}\right)^2}\right], \tag{12}$$

and

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{\alpha\, n}{\lambda^2} - (n-D)\left(\frac{x_D}{\lambda}\right)^{\alpha}\left[\frac{\ln\lambda}{1+\left(\frac{x_D}{\lambda}\right)^{\alpha}}\right]^2 - 2\sum_{i=1}^{D}\left(\frac{x_i}{\lambda}\right)^{\alpha}\left[\frac{\ln\lambda}{1+\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right]^2. \tag{13}$$

After that, $\Lambda = \Lambda(\alpha,\lambda)$, the Fisher information matrix, is derived by subtracting the expectations of Eqs.(11 - 13). A bivariate normal with mean $(\alpha\,\lambda)$ and covariance matrix $\Lambda^{-1} = \Lambda^{-1}(\alpha,\lambda)$ is roughly represented by $(\hat{\alpha},\ \hat{\lambda})$ under some moderate regularity criteria. In practice, we usually estimate $\Lambda^{-1}(\alpha,\lambda)$ by $\Lambda^{-1}(\hat{\alpha},\ \hat{\lambda})$. An alternative and equally valid approach is to employ the following approximation;

$$(\hat{\alpha},\ \hat{\lambda}) \sim N\left[(\alpha,\lambda),\ \Lambda^{-1}(\hat{\alpha},\ \hat{\lambda})\right],$$

where the variance-covariance matrix of the unknown parameters, Λ^{-1}, can be derived by

$$\Lambda^{-1}(\ \hat{\alpha},\ \hat{\lambda}\ ) = \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \alpha^2} & -\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \ell}{\partial \lambda^2} \end{bmatrix}^{-1}_{(\hat{\alpha},\hat{\lambda})} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

The $100(1-\gamma)\%$ normal approximate confidence intervals (CIs) for α and λ can be formulated as follows:

$$\begin{cases} \hat{\alpha} \mp Z_{\gamma/2}\sqrt{\Lambda_{11}} \\ \hat{\lambda} \mp Z_{\gamma/2}\sqrt{\Lambda_{22}} \end{cases} \tag{14}$$

where the right-tail probability $\gamma/2$ is represented by the percentile $Z_{\gamma/2}$ of the standard normal distribution.

In addition, finding the variances of the reliability and hazard functions is crucial to establishing the asymptotic confidence intervals (CIs) for them. To approximate the variances of $\hat{R}(t)$ and $\hat{H}(t)$, we employ the delta method. The delta method is a general technique used to compute CIs for functions of maximum likelihood estimators (MLEs). It entails building a linear approximation of the complex function, which can be used for large sample inference, and then calculating the variance of this reduced linear function. For more details, refer to Ref. [22]. Let

$$'\mathscr{G}_1 = \left(\frac{\partial R(t)}{\partial \alpha}, \frac{\partial R(t)}{\partial \lambda}\right) \ and\ '\mathscr{G}_2 = \left(\frac{\partial H(t)}{\partial \alpha}, \frac{\partial H(t)}{\partial \lambda}\right),$$

where

$$\begin{cases} \frac{\partial R(t)}{\partial \alpha} = \frac{2\left(\frac{\lambda}{t}\right)^{\alpha} ln\left(\frac{\lambda}{t}\right)}{\left(1+\left(\frac{\lambda}{t}\right)^{\alpha}\right)^2} \\ \frac{\partial R(t)}{\partial \lambda} = \frac{2\left(\frac{\alpha}{t}\right)\left(\frac{\lambda}{t}\right)^{\alpha-1}}{\left(1+\left(\frac{\lambda}{t}\right)^{\alpha}\right)^2} \end{cases}$$

and

$$\begin{cases} \frac{\partial H(t)}{\partial \alpha} = \frac{1+\left(\frac{\lambda}{t}\right)^{\alpha}\left(1-\alpha\ ln\left(\frac{\lambda}{t}\right)\right)}{t\left(1+\left(\frac{\lambda}{t}\right)^{\alpha}\right)^2} \\ \frac{\partial H(t)}{\partial \lambda} = \frac{-\ \alpha^2\ \left(\frac{\lambda}{t}\right)^{\alpha-1}}{\left(t\left(1+\left(\frac{\lambda}{t}\right)^{\alpha}\right)\right)^2} \end{cases}$$

Then the approximate estimates of $\mathrm{Var}(\hat{R}(t))$ and $\mathrm{Var}(\hat{H}(t))$ are given, respectively, by

$$\begin{cases} \widehat{Var}(\hat{R}) \simeq \left[{}'\mathscr{G}_1\ \Lambda^{-1}\ \mathscr{G}_1\right]_{(\ \hat{\alpha},\ \hat{\lambda}\ )} \\ \widehat{Var}(\hat{H}) \simeq \left[{}'\mathscr{G}_2\ \Lambda^{-1}\ \mathscr{G}_2\right]_{(\ \hat{\alpha},\ \hat{\lambda}\ )} \end{cases} \tag{15}$$

Thus, both

$$\frac{(\hat{R}(t)-R(t))}{\sqrt{\widehat{Var}(\hat{R})}} \quad and \quad \frac{(\hat{H}(t)-H(t))}{\sqrt{\widehat{Var}(\hat{H})}} \quad \sim \quad N(0,1) \tag{16}$$

are asymptotically. The approximate CIs for $R(t)$ and $H(t)$ are obtained from these results as

$$\begin{cases} \hat{R}(t) \mp Z_{\gamma/2}\sqrt{\widehat{Var}(\hat{R})} \\ \hat{H}(t) \mp Z_{\gamma/2}\sqrt{\widehat{Var}(\hat{H})} \end{cases} \tag{17}$$

3 Bootstrap confidence intervals

While asymptotic confidence interval approaches are based on the law of large numbers, it is important to remember that the sample size is frequently insufficient in many real-world scenarios. It is important to remember that these techniques have limits when used to small sample sets. Nevertheless, Ref. [23] presented the bootstrap method as a substitute technique for constructing confidence intervals in order to overcome this problem. . We introduce the parametric Boot-p technique, which builds the bootstrap confidence intervals for α, λ, $R(t)$, and $H(t)$. Algorithm (1) details the Boot-p method's step-by-step implementation.

Algorithm 1. Boot-P method.

Step 1: Compute the MLEs of α and λ, using GHT1CS $X_1, X_2, \ldots, X_D$.

Step 2: Generate a bootstrap sample (say $X_1^*, X_2^*, \ldots, X_D^*$) from NP$(\hat{\alpha}, \hat{\lambda})$.

Step 3: Calculate the bootstrap samples $\alpha^*, \lambda^*, R^*(t)$ and $H^*(t)$ based on $X_1^*, X_2^*, \ldots, X_D^*$.

Step 4: Redo Steps $2-3$ N$_{boot}$ time.

Step 5: Set the bootstrap samples in ascending order as

$$\begin{cases} (\alpha^*_{(1)}, \alpha^*_{(2)}, \ldots, \alpha_{(Nboot)^*}) \\ (\lambda^*_{(1)}, \lambda^*_{(2)}, \ldots, \lambda_{(Nboot)^*}) \end{cases}$$

and

$$\begin{cases} (R^*_{(1)}(t), R^*_{(2)}(t), \ldots, R_{(Nboot)^*}(t)) \\ (H^*_{(1)}(t), H^*_{(2)}(t), \ldots, H_{(Nboot)^*}(t)) \end{cases}$$

Step 6: The $100(1-\gamma)\%$ boot-p intervals of $\alpha, \lambda, R(t)$ and $H(t)$ (say Φ) are given, respectively, by

$$\left(\hat{\Phi}^*_{[N_{boot}(\frac{\gamma}{2})]} \quad , \quad \hat{\Phi}^*_{[N_{boot}(1-\frac{\gamma}{2})]}\right)$$

4 Bayesian estimation

The Bayesian perspective is a potent and legitimate substitute for classical estimation that has garnered considerable interest in statistical inference. Its ability to incorporate prior information in the analysis makes it highly advantageous in areas such as reliability, lifetime studies, and related fields, where limited data availability presents a significant obstacle. This section presents a method for Bayesian inference that uses the Markov Chain Monte Carlo (MCMC) methodology to estimate the hazard rate function (HRF), reliability function (RF), and parameters α and λ. Use of the squared error (SE) loss function guides the estimate process. We also use the MCMC approach to create matching Credible Intervals (CIs). We consider the following joint prior PDF, given by

$$P(\alpha,\lambda) \propto \alpha^{v}\lambda^{\alpha b-1}c^{-\alpha}, \; \alpha > 0, \; 0 < \lambda < d. \tag{18}$$

Where $v, b, c, d > 0$ and $d^b < c$. The prior distribution mentioned in this context was originally introduced by Ref. [24] and subsequently generalized by Refs. [25, 26]. This prior specifies α to follow a gamma distribution with parameters v and $(\ln c - b\ln d)$, while $P(\lambda|\alpha)$ is modeled as a power function distribution.

$$P(\lambda|\alpha) \propto b\alpha\lambda^{\alpha b-1} \; , \; 0 < \lambda < d.$$

To represent the posterior distribution of α, λ, and the data, we combine the likelihood function given in Eq.(6) with the joint prior distribution given in Eq. (18).

$$P^*(\alpha,\lambda) = k^{-1}\alpha^{v+D}\lambda^{\alpha(D+b)-1}c^{-\alpha}\left(1+\left(\frac{x_D}{\lambda}\right)^{\alpha}\right)^{(D-n)} \times \prod_{i=1}^{D}\frac{x_i^{-(\alpha+1)}}{\left(1+\left(\frac{\lambda}{x_i}\right)^{\alpha}\right)^2} \tag{19}$$

The normalized constant K is determined as follows:

$$K = \int_0^{\infty}\int_0^{\infty}\alpha^{v+D}\lambda^{\alpha(D+b)-1}\,c^{-\alpha}\left(1+\left(\frac{x_D}{\lambda}\right)^{\alpha}\right)^{(D-n)} \times \prod_{i=1}^{D}\frac{x_i^{-(\alpha+1)}}{\left(1+\left(\frac{\lambda}{x_i}\right)^{\alpha}\right)^2}\,d\alpha\,d\lambda$$

The Bayes estimator (BE) for any function $\psi(\alpha,\lambda)$ can be stated as follows under the squared error loss function:

$$\begin{aligned}\tilde{\psi}(\alpha,\lambda) &= E_{\alpha,\,\lambda|X}\,[\psi(\alpha,\lambda)] \\ &= \int_0^{\infty}\int_0^{\infty}P^*(\alpha,\lambda)\,\psi(\alpha,\lambda)\,d\alpha\,d\lambda\end{aligned} \tag{20}$$

Considering Eq.(19), it is apparent that the analytical investigation of α and λ alone is insufficient for obtaining the Bayes (or BCI) estimators of α, λ, $R(t)$, and $H(t)$. Therefore, following the approach presented in Ref. [27], we suggest utilizing the

Metropolis-Hastings within Gibbs (MH-G) sampler. To facilitate this, it is necessary to derive the probability density functions (PDFs) of α and λ as follows:

$$P_1^*(\alpha|X) \propto \alpha^{(\nu+\frac{2D}{3}+1)-1} exp(-\alpha(\ln c + \sum_{i=1}^{D} \ln x_i)) \tag{21}$$

$$P_2^*(\lambda|\alpha,X) \propto \lambda^{\alpha(D+b)-1}\left(1+\left(\frac{x_D}{\lambda}\right)^{\alpha}\right)^{(2(D\;-\;n))/3} \times \prod_{i=1}^{D} \frac{1}{x_i\left(1+\left(\frac{\lambda}{x_i}\right)^{\alpha}\right)^{7/2}} \tag{22}$$

and

$$q(\alpha,\lambda) \propto \alpha^{\frac{D}{3}}\left(1+\left(\frac{x_D}{\lambda}\right)^{\alpha}\right)^{\frac{D-n}{3}} \prod_{i=1}^{D} \frac{1}{x_i\left(1+\left(\frac{\lambda}{x_i}\right)^{\alpha}\right)^{-3/2}}$$

We consider the density function $P_1^*(\alpha|X)$, which is a Gamma distribution with a shape parameter of $(\nu + \frac{2D}{3} + 1)$ and a scale parameter of $(\ln c + \sum_{i=1}^{D} \ln x_i)$. Employing a Gamma-generating algorithm facilitates easy sampling of α. Additionally, there is no analytical way to reduce the conditional posterior density of λ to a well-known distribution. To address this problem, we employ the Metropolis-Hastings (M-H) algorithm within Gibbs sampling to conduct the Markov Chain Monte Carlo (MCMC) methodology. For further details, please refer to the Ref. [28].

Algorithm 2. Bayesian estimates under square loss function.

Step 1: $\hat{\alpha}^{(0)}, \hat{\lambda}^{(0)}$ represents the initial value to be used, which is the MLEs.

Step 2: Set $i = 1$.

Step 3: Generate $\alpha^{(i)}$ from $\text{Gamma}\left(\nu + \frac{2D}{3} + 1\;,\; \ln c + \sum_{i=1}^{D} \ln x_i\right)$.

Step 4: Create $\lambda^{(i)}$ using the M-H algorithm from $P_2^*\left(\lambda^{(i-1)}|\alpha^{(i)},X\right)$, respectively, with the normal distribution $N\left(\lambda^{(i-1)}, var(\hat{\lambda})\right)$.

Step 5: Compute $R(t)$ and $H(t)$

$$\begin{cases} R^{(i)}(t) = \dfrac{2\left(\frac{\lambda^{(i)}}{t}\right)^{\alpha^{(i)}}}{1+\left(\frac{\lambda^{(i)}}{t}\right)^{\alpha^{(i)}}} \\ H^{(i)}(t) = \dfrac{\alpha^{(i)}}{t\left(1+\left(\frac{\lambda^{(i)}}{t}\right)^{\alpha^{(i)}}\right)} \end{cases}$$

Step 6: Repeat Steps (3) through (5) M times, setting $i = i + 1$.

Step 7: For generated $(\alpha^{(i)}, \lambda^{(i)})$, calculate $q(\alpha^{(i)}, \lambda^{(i)})$ and the importance weight (say $weight^*$), $i = 1,2,\ldots,M$.

$$weight^*_{(i)} = \frac{q_{(i)}}{\sum_{i=M_0+1}^{M} q_{(i)}}$$

In order to eliminate the impact of initial value selection and ensure convergence, we discard the first M_0 simulated varieties. Subsequently, a selected sample is formed, consisting of $\alpha^{(i)}$, $\lambda^{(i)}$, $R^{(i)}(t)$, and $h^{(i)}(t)$ for $i = M_0+1,\ldots,M$, where M is sufficiently large. The chosen sample can be used to construct Bayesian inference as an approximate posterior sample.

Step 8: It is feasible to obtain the approximate Bayesian estimate of ζ (where $\zeta = (\alpha, \lambda, R(t), andH(t))$) by utilising the squared error loss function (SELF).

$$\hat{\zeta} = \sum_{i=M_0+1}^{M} weight^{*(i)}\, \zeta^{(i)}$$

where M_0 is the burn-in period and

$$\zeta^{(i)} = \alpha^{(i)}, \lambda^{(i)}, R^{(i)}(t) and H^{(i)}(t)$$

Now, using the Metropolis-Hastings (M-H) algorithm within the Gibbs sampling technique, we create the $100(1-\gamma)\%$ credible interval of ζ, (where $\zeta = \alpha, \lambda, R(t)$ and $H(t)$) in the same manner as stated in Ref. [29]. Suppose $\zeta_{\mathscr{P}}$ is such that $P[\zeta \le \zeta_{\mathscr{P}}|x] = \mathscr{P}$, for $0 < \mathscr{P} < 1$. Considering the following function

$$K(\zeta) = \begin{cases} 0 & if \quad \zeta > \zeta_{\mathscr{P}} \\ 1 & if \quad \zeta \le \zeta_{\mathscr{P}} \end{cases}$$

Such that $E[k(\zeta|x)] = \mathscr{P}$. Thus, the created sample can be used to derive a simulation consistent Bayes estimate of $\zeta_{\mathscr{P}}$ under SELF. $\{\zeta_{M_0+1}, \zeta_{M_0+2}, \ldots, \zeta_M\}$

as follows. Let $\zeta_{M_0+1} = \zeta(\alpha_{M_0+1}, \lambda_{M_0+1}, R_{M_0+1}(t), h_{M_0+1}(t)), \ldots, \zeta_M = \zeta(\alpha_M, \lambda_M, R_M(t), h_M(t))$ and rearrange $(\zeta_{(M_0+1)}, weight^*_{(M_0+1)}), \ldots, (\zeta_{(M)}, weight^*_{(M)})$ as follows $(\zeta_{[M_0+1]}, weight^*_{[M_0+1]}), \ldots, (\zeta_{[M]}, weight^*_{[M]})$. Take note that $\zeta_{[i]}$ and $weight^*_{[i]}$ are related, but they are not ordered. For $\zeta_{\mathscr{P}}$, a simulation-consistent Bayes estimate can be produced as $\hat{\zeta}_{\mathscr{P}} = \zeta_{(M_{\mathscr{P}})}$ where,

$$\sum_{i=M_0+1}^{M_{\mathscr{P}}} weight^*_{[i]} \leq \mathscr{P} < \sum_{i=M_0+1}^{M_{\mathscr{P}}+1} weight^*_{[i]}$$

Now, a $100(1-\gamma)\%$ credible interval can be obtained as follows by using the aforementioned procedure:

$$\left(\hat{\zeta}_{\mathscr{P}^*}\ ,\ \hat{\zeta}_{\mathscr{P}^*+1-\gamma}\right) \tag{23}$$

Where

$$\mathscr{P}^* = weight^*_{[M_0+1]}, weight^*_{[M_0+1]} + weight^*_{[M_0+2]}, \ldots, \sum_{i=M_0+1}^{M_{1-\gamma}} weight^*_{[i]}$$

5 Numerical illustration

This section includes in-depth simulation experiments that were carried out to evaluate the efficacy of the suggested techniques. Furthermore, a real-world example is provided to illustrate their usefulness.

5.1 Computational simulations

the acquired classical and Bayesian posterior estimators of $\alpha, \lambda, R(t)$, and $H(t)$ should be tested for behaviour . Utilizing various choices of $(n, , (m,k), \tau)$, large 1000 GHT1CS were collected from NP$(1.0, 1.0)$. From $t = 1.3$, the acquired estimates of $R(t)$ and $H(t)$ were evaluated when their true values were 0.8696 and 0.4348, respectively. For each combination of $\tau = 5.0, 7.0$ and $n = 30, 50, 80$ with three sets of fixed numbers $(k,m) = (15,20), (15,25), (30,35), (30,40), (40,50), (40,60)$. To obtain an GHT1CS sample from the NP distribution, after fixing $(n, (k,m), \tau)$, perform the following algorithm (3):

Algorithm 3. GHTC Samples.

Step 1: Create n independent variables $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ from U$(0,1)$ distribution.

Step 2: For given k, m, τ, set D as,

$$D = \begin{cases} k & if \quad \tau < k \\ m & if \quad k \leq \tau < m \\ d & if \quad \tau \geq m \end{cases}$$

Step 3: GHTCS The inverse function approach is used to generate; $X = (X_{1:n}, X_{2:n}, \ldots, X_{D:n})$; $X = \lambda \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{-1/\alpha}$

For each GHTCS, the estimates of α, λ, $R(t)$, and $h(t)$ were evaluated using both the likelihood and bootstrap techniques in the frequentist viewpoint. ACIs, or approximate confidence intervals, were also calculated for these estimations at 95%. Two sets of hyper-parameters were employed to examine how the priors affected the parameters α and λ. These were as follows: Prior-1 (non-informative prior); $\nu = -1, b = 0, c = 1, d \longrightarrow \infty$, and Prior- 2; $\nu = 0.02, b = 2, c = 5, d = 2$. To estimate each parameter, we utilized the MH-G algorithm outlined in Section 4, collecting a total of 11,000 Markovian iterations. The initial 1,000 iterations were disregarded to mitigate the impact of the initial values. Subsequently, from the remaining 10,000 iterations, we obtained Bayes estimates for α, λ, $R(t)$, and $H(t)$ using the likelihood approach. Additionally, we computed 95% Bayesian credible intervals (BCIs) for each estimate.

The point estimates of α obtained were evaluated and compared using two criteria, namely:

1. Average estimates(AEs)

$$AE(\alpha^*) = \frac{1}{\mathscr{D}} \sum_{i=1}^{\mathscr{D}} \alpha^{*(i)}$$

2. Mean-squared errors (MSEs)

$$MSE(\alpha^*) = \frac{1}{\mathscr{D}} \sum_{i=1}^{\mathscr{D}} (\alpha^{*(i)} - \alpha)^2$$

The number of replications is represented by $\mathscr{D}$, and the estimate of α at the i-th sample is indicated by $\alpha^{*(i)}$.

Additionally, two criteria known as were used to compare the performance of the derived interval estimates:

1. Average confidence lengths (ACLs), delivered as

$$ACI_{(1-\gamma)\%}(\alpha) = \frac{1}{\mathscr{D}} \sum_{i=1}^{\mathscr{D}} (\mathscr{U}_{\alpha^{*(i)}} - \mathscr{L}_{\alpha^{*(i)}})$$

2. The percentages of coverage (CPs), provided as

$$CP_{(1-\gamma)\%}(\alpha) = \frac{1}{\mathscr{D}} \sum_{i=1}^{\mathscr{D}} \mathbf{1}_{(\mathscr{U}_{\alpha^{*(i)}}, \mathscr{L}_{\alpha^{*(i)}})}(\alpha)$$

correspondingly, where the indicator operator is indicated by $\mathbf{1}_{(.)}$. Similarly, it is easy to derive the values of λ, $R(t)$, and $H(t)$ as well as the simulated absolute error (AE), mean squared error (MSE), average coverage length (ACL), and coverage probability (CP). Tables (1 - 2) display the estimates' AE and MSE findings, whereas Tables (3 - 4) display the estimates' ACL and CP results. There are other analyses of how well the suggested inferential approaches perform that are covered. . Based on Tables (1 - 4), the following observations are listed:

1-All the estimated values of α, λ, $R(t)$, and $H(t)$ exhibited satisfactory performance in terms of minimal absolute errors (AEs), mean squared errors (MSEs), and average coverage lengths (ACLs), while achieving the highest coverage probabilities (CPs).

2-With an increase in the value of n, the obtained point or interval estimates exhibited improved accuracy and reliability.

3-In comparison to Prior-1, all Bayesian results obtained using Prior-2 demonstrated superior performance. Since Prior-2's variance was less than Prior-1's, this result was expected.

4-Coverage probabilities (CPs) of the estimated asymptotic or Bayesian intervals for α, λ, $R(t)$, or $H(t)$ were near the designated nominal level for most cases..

5-Comparing the point estimation techniques, it was found that most of the tests:

i-The estimates of λ and α from the Bootstrap, together with those of $R(t)$ and $H(t)$ from the likelihood technique, all performed satisfactorily in a classical scenario when compared to the others.

ii-In a Bayesian approach, the values of α, λ, $R(t)$, and $H(t)$ are inferred from the likelihood.

6-In the majority of the tests, a comparison of the interval estimation approaches revealed that:

i-The ACIs of $\alpha, \lambda, R(t)$, and $H(t)$ derived from the Bootstrap outperformed the others in a traditional setting.

ii-The BCIs of α derived from the likelihood function and those of $\lambda, R(t)$, and $H(t)$ derived from the Bootstrap outperformed the rest in a Bayes setup.

7-Therefore, it is advised to estimate the NP's life parameters using the Bayes technique via the MH-G algorithm based on the GHTCS plan.

Table 1. Average estimates and MSEs of MLE, Boot-p and Bayesian methods for (α, λ, R, H) under SEL function with Prior-1.

n	(k,m)	τ	Par	MLE		Boot-p		Bayesian	
				AE	MSE	AE	MSE	AE	MSE
30	(15,20)	5.0	$\hat{\alpha}$	1.086	0.239	1.271	0.381	0.927	0.301
			$\hat{\lambda}$	1.189	0.136	1.266	0.195	1.212	0.317
			$\hat{R}(t)$	0.947	0.080	0.965	0.092	0.925	0.114
			$\hat{H}(t)$	0.469	0.099	0.685	0.157	0.378	0.187
		7.0	$\hat{\alpha}$	1.169	0.221	1.185	0.288	0.898	0.297
			$\hat{\lambda}$	1.169	0.113	1.149	0.178	1.143	0.299
			$\hat{R}(t)$	0.995	0.079	0.989	0.090	0.899	0.135
			$\hat{H}(t)$	0.555	0.097	0.695	0.141	0.385	0.152
	(15,25)	5.0	$\hat{\alpha}$	1.066	0.209	1.171	0.281	0.827	0.281
			$\hat{\lambda}$	1.089	0.106	1.166	0.175	1.112	0.207
			$\hat{R}(t)$	0.907	0.076	0.935	0.072	0.925	0.114
			$\hat{H}(t)$	0.449	0.096	0.485	0.127	0.338	0.107
		7.0	$\hat{\alpha}$	1.069	0.201	1.185	0.268	0.798	0.177
			$\hat{\lambda}$	1.069	0.093	1.139	0.148	1.043	0.159
			$\hat{R}(t)$	0.895	0.039	0.919	0.060	0.889	0.108
			$\hat{H}(t)$	0.455	0.094	0.495	0.118	0.345	0.132
50	(30,35)	5.0	$\hat{\alpha}$	1.106	0.169	1.182	0.231	0.982	0.183
			$\hat{\lambda}$	1.136	0.090	1.092	0.098	1.223	0.167
			$\hat{R}(t)$	0.893	0.061	0.895	0.047	0.999	0.093
			$\hat{H}(t)$	0.499	0.092	0.563	0.114	0.396	0.098
		7.0	$\hat{\alpha}$	1.120	0.099	1.180	0.205	0.894	0.171
			$\hat{\lambda}$	1.133	0.097	1.086	0.094	1.092	0.138
			$\hat{R}(t)$	0.893	0.064	0.877	0.044	0.859	0.056
			$\hat{H}(t)$	0.478	0.090	0.479	0.108	0.388	0.083
	(30,40)	5.0	$\hat{\alpha}$	1.126	0.159	1.192	0.141	0.912	0.143
			$\hat{\lambda}$	1.036	0.048	1.072	0.078	1.123	0.157
			$\hat{R}(t)$	0.873	0.021	0.885	0.037	0.929	0.033
			$\hat{H}(t)$	0.489	0.089	0.513	0.104	0.376	0.078
		7.0	$\hat{\alpha}$	1.020	0.097	1.080	0.125	0.864	0.111
			$\hat{\lambda}$	1.033	0.047	1.076	0.054	1.062	0.113
			$\hat{R}(t)$	0.883	0.024	0.897	0.035	0.899	0.026
			$\hat{H}(t)$	0.438	0.085	0.459	0.101	0.368	0.034

Table 2. Averages and MSEs of MLE, Boot-p and Bayesian methods for (α, λ, R, H) under SEL function with Prior-2.

n	(k,m)	τ	Par	MLE AE	MLE MSE	Boot-p AE	Boot-p MSE	Bayesian AE	Bayesian MSE
30	$(15,20)$	5.0	$\hat{\alpha}$	1.097	0.211	1.194	0.285	1.170	0.239
			$\hat{\lambda}$	1.064	0.099	1.121	0.183	1.194	0.266
			$\hat{R}(t)$	0.894	0.093	0.948	0.091	0.948	0.142
			$\hat{H}(t)$	0.473	0.098	0.599	0.131	0.375	0.148
		7.0	$\hat{\alpha}$	1.126	0.184	1.237	0.275	1.151	0.209
			$\hat{\lambda}$	1.078	0.093	1.163	0.171	1.143	0.224
			$\hat{R}(t)$	0.895	0.089	0.959	0.078	0.964	0.139
			$\hat{H}(t)$	0.499	0.093	0.597	0.123	0.366	0.126
	$(15,25)$	5.0	$\hat{\alpha}$	1.077	0.181	1.184	0.265	0.850	0.200
			$\hat{\lambda}$	1.044	0.086	1.111	0.143	1.184	0.156
			$\hat{R}(t)$	0.884	0.079	0.908	0.051	0.928	0.122
			$\hat{H}(t)$	0.463	0.088	0.499	0.111	0.355	0.118
		7.0	$\hat{\alpha}$	1.116	0.180	1.227	0.255	0.841	0.179
			$\hat{\lambda}$	1.058	0.071	1.123	0.131	1.103	0.114
			$\hat{R}(t)$	0.885	0.063	0.909	0.048	0.924	0.079
			$\hat{H}(t)$	0.479	0.073	0.517	0.104	0.346	0.106
50	$(30,35)$	5.0	$\hat{\alpha}$	1.176	0.170	1.208	0.249	0.946	0.147
			$\hat{\lambda}$	1.067	0.037	1.068	0.089	1.167	0.109
			$\hat{R}(t)$	0.869	0.056	0.889	0.042	0.966	0.072
			$\hat{H}(t)$	0.499	0.060	0.526	0.063	0.396	0.087
		7.0	$\hat{\alpha}$	1.069	0.162	1.074	0.174	0.792	0.132
			$\hat{\lambda}$	1.065	0.068	1.077	0.077	0.998	0.106
			$\hat{R}(t)$	0.878	0.050	0.897	0.037	0.887	0.068
			$\hat{H}(t)$	0.436	0.056	0.468	0.050	0.368	0.065
	$(30,40)$	5.0	$\hat{\alpha}$	1.136	0.160	1.203	0.249	0.943	0.122
			$\hat{\lambda}$	1.027	0.032	1.065	0.068	1.157	0.101
			$\hat{R}(t)$	0.866	0.043	0.879	0.026	0.936	0.056
			$\hat{H}(t)$	0.497	0.094	0.521	0.043	0.386	0.053
		7.0	$\hat{\alpha}$	1.019	0.146	1.073	0.124	0.782	0.103
			$\hat{\lambda}$	1.025	0.028	1.067	0.060	0.993	0.069
			$\hat{R}(t)$	0.879	0.020	0.894	0.019	0.884	0.048
			$\hat{H}(t)$	0.431	0.040	0.458	0.034	0.338	0.045

Table 1. continued.

n	(k,m)	τ	Par	MLE AE	MLE MSE	Boot-p AE	Boot-p MSE	Bayesian AE	Bayesian MSE
80	$(40,50)$	5.0	$\hat{\alpha}$	1.028	0.115	1.076	0.094	0.996	0.107
			$\hat{\lambda}$	1.036	0.045	1.077	0.040	1.099	0.106
			$\hat{R}(t)$	0.896	0.020	0.869	0.032	0.946	0.017
			$\hat{H}(t)$	0.457	0.079	0.491	0.096	0.439	0.029
		7.0	$\hat{\alpha}$	0.996	0.077	1.088	0.110	0.855	0.097
			$\hat{\lambda}$	1.079	0.040	1.095	0.037	1.068	0.099
			$\hat{R}(t)$	0.999	0.018	0.899	0.029	0.799	0.015
			$\hat{H}(t)$	0.466	0.069	0.478	0.059	0.349	0.023
	$(40,60)$	5.0	$\hat{\alpha}$	1.006	0.059	1.051	0.080	0.972	0.073
			$\hat{\lambda}$	1.022	0.031	1.047	0.032	1.099	0.063
			$\hat{R}(t)$	0.871	0.015	0.887	0.020	0.912	0.011
			$\hat{H}(t)$	0.434	0.057	0.451	0.033	0.409	0.018
		7.0	$\hat{\alpha}$	0.998	0.042	1.038	0.072	0.813	0.053
			$\hat{\lambda}$	1.019	0.026	1.045	0.029	1.066	0.042
			$\hat{R}(t)$	0.879	0.011	0.887	0.014	0.896	0.009
			$\hat{H}(t)$	0.431	0.049	0.445	0.022	0.465	0.012

Table 2. continued.

n	(k,m)	τ	Par	MLE AE	MLE MSE	Boot-p AE	Boot-p MSE	Bayesian AE	Bayesian MSE
80	$(40,50)$	5.0	$\hat{\alpha}$	0.985	0.118	0.958	0.120	0.967	0.101
			$\hat{\lambda}$	1.015	0.024	1.022	0.076	1.107	0.055
			$\hat{R}(t)$	0.891	0.016	0.869	0.017	0.935	0.046
			$\hat{H}(t)$	0.408	0.036	0.424	0.032	0.463	0.035
		7.0	$\hat{\alpha}$	0.974	0.108	1.006	0.104	0.899	0.087
			$\hat{\lambda}$	1.023	0.016	1.034	0.056	1.007	0.045
			$\hat{R}(t)$	0.885	0.013	0.853	0.012	0.936	0.033
			$\hat{H}(t)$	0.418	0.030	0.472	0.016	0.435	0.022
	$(40,60)$	5.0	$\hat{\alpha}$	0.955	0.088	0.998	0.080	0.929	0.056
			$\hat{\lambda}$	1.035	0.006	1.062	0.076	1.303	0.025
			$\hat{R}(t)$	0.891	0.001	0.899	0.009	0.985	0.146
			$\hat{H}(t)$	0.408	0.026	0.424	0.002	0.363	0.015
		7.0	$\hat{\alpha}$	0.974	0.058	1.010	0.060	0.859	0.017
			$\hat{\lambda}$	1.003	0.009	1.051	0.056	1.037	0.011
			$\hat{R}(t)$	0.885	0.008	0.894	0.003	0.900	0.043
			$\hat{H}(t)$	0.418	0.020	0.431	0.006	0.365	0.008

5.2 Illustrative examples

To investigate the practical applicability of the theoretical results for α, λ, $R(t)$, and $H(t)$, a real dataset called the Plane 720 data was utilized. The aircraft's air conditioning system failure times (in 10 hours) are represented by the Plane 720 data. This dataset, which is widely known, has been previously studied in works such as Refs. [30, 31]. The detailed information of the Plane 720 data is provided in Table (5) below:

Table 3. ALs and CPs of MLE, Boot-p and Bayesian methods for (α, λ, R, H) under SEL function with Prior-1.

n	(k,m)	τ	Par	MLE AL	MLE CP	Boot-p AL	Boot-p CP	Bayesian AL	Bayesian CP
30	(15,20)	5.0	$\hat{\alpha}$	1.764	0.883	0.899	0.896	0.966	0.824
			$\hat{\lambda}$	1.749	0.931	1.929	0.923	1.555	0.903
			$\hat{R}(t)$	0.763	0.924	0.669	0.945	0.537	0.904
			$\hat{H}(t)$	0.496	0.936	0.399	0.949	0.437	0.904
		7.0	$\hat{\alpha}$	1.935	0.865	0.899	0.921	0.898	0.933
			$\hat{\lambda}$	1.941	0.923	1.697	0.925	1.430	0.903
			$\hat{R}(t)$	0.693	0.906	0.598	0.906	0.567	0.997
			$\hat{H}(t)$	0.469	0.913	0.694	0.974	0.631	0.986
	(15,25)	5.0	$\hat{\alpha}$	1.664	0.882	0.895	0.894	0.926	0.924
			$\hat{\lambda}$	1.742	0.911	1.926	0.922	1.525	0.933
			$\hat{R}(t)$	0.713	0.944	0.619	0.955	0.567	0.964
			$\hat{H}(t)$	0.480	0.926	0.3965	0.946	0.427	0.964
		7.0	$\hat{\alpha}$	1.435	0.895	0.889	0.901	0.888	0.923
			$\hat{\lambda}$	1.541	0.903	1.657	0.915	1.450	0.973
			$\hat{R}(t)$	0.633	0.946	0.558	0.956	0.547	0.967
			$\hat{H}(t)$	0.439	0.903	0.394	0.934	0.431	0.946
50	(30,35)	5.0	$\hat{\alpha}$	0.956	0.866	0.699	0.914	0.696	0.906
			$\hat{\lambda}$	1.281	0.905	1.139	0.906	1.099	0.926
			$\hat{R}(t)$	0.491	0.996	0.564	0.969	0.467	0.978
			$\hat{H}(t)$	0.368	0.962	0.296	0.936	0.326	0.966
		7.0	$\hat{\alpha}$	0.959	0.893	0.669	0.902	0.696	0.927
			$\hat{\lambda}$	1.089	0.906	1.169	0.906	1.089	0.918
			$\hat{R}(t)$	0.641	0.906	0.905	0.907	0.497	0.984
			$\hat{H}(t)$	0.328	0.902	0.682	0.927	0.328	0.906
	(30,40)	5.0	$\hat{\alpha}$	0.954	0.896	0.629	0.904	0.636	0.936
			$\hat{\lambda}$	1.081	0.925	1.109	0.936	1.029	0.946
			$\hat{R}(t)$	0.441	0.956	0.504	0.964	0.417	0.975
			$\hat{H}(t)$	0.308	0.912	0.292	0.934	0.321	0.946
		7.0	$\hat{\alpha}$	0.954	0.899	0.629	0.912	0.636	0.923
			$\hat{\lambda}$	1.081	0.916	1.109	0.926	1.029	0.938
			$\hat{R}(t)$	0.541	0.936	0.605	0.947	0.417	0.964
			$\hat{H}(t)$	0.308	0.912	0.292	0.923	0.321	0.946

Table 4. ALs and CPs of MLE, Boot-p and Bayesian methods for (α, λ, R, H) under SEL function with Prior-2.

n	(k,m)	τ	Par	MLE AL	MLE CP	Boot-p AL	Boot-p CP	Bayesian AL	Bayesian CP
30	(15,20)	5.0	$\hat{\alpha}$	1.902	0.993	0.895	0.924	0.899	0.904
			$\hat{\lambda}$	1.666	0.912	1.996	0.934	1.794	0.968
			$\hat{R}(t)$	0.990	0.965	0.662	0.967	0.936	0.996
			$\hat{H}(t)$	0.597	0.912	0.908	0.916	0.697	0.906
		7.0	$\hat{\alpha}$	1.953	0.913	0.914	0.964	0.871	0.907
			$\hat{\lambda}$	1.436	0.932	1.989	0.946	1.478	0.996
			$\hat{R}(t)$	0.667	0.976	0.681	0.901	0.587	0.908
			$\hat{H}(t)$	0.491	0.962	0.469	0.904	0.903	0.905
	(15,25)	5.0	$\hat{\alpha}$	1.702	0.893	0.885	0.904	0.891	0.944
			$\hat{\lambda}$	1.656	0.902	1.796	0.914	1.594	0.948
			$\hat{R}(t)$	0.690	0.935	0.622	0.947	0.536	0.976
			$\hat{H}(t)$	0.517	0.902	0.408	0.906	0.397	0.956
		7.0	$\hat{\alpha}$	1.453	0.903	0.904	0.914	0.821	0.947
			$\hat{\lambda}$	1.432	0.912	1.489	0.942	1.428	0.976
			$\hat{R}(t)$	0.627	0.956	0.661	0.966	0.527	0.968
			$\hat{H}(t)$	0.451	0.912	0.419	0.934	0.933	0.965
50	(30,35)	5.0	$\hat{\alpha}$	1.399	0.926	0.691	0.902	0.798	0.906
			$\hat{\lambda}$	1.397	0.904	1.104	0.959	0.962	0.916
			$\hat{R}(t)$	0.592	0.906	0.599	0.904	0.618	0.926
			$\hat{H}(t)$	0.687	0.834	0.390	0.928	0.399	0.906
		7.0	$\hat{\alpha}$	0.916	0.926	0.984	0.903	0.985	0.905
			$\hat{\lambda}$	1.242	0.905	1.764	0.924	1.094	0.934
			$\hat{R}(t)$	0.468	0.904	0.493	0.914	0.498	0.917
			$\hat{H}(t)$	0.497	0.917	0.674	0.914	0.388	0.916
	(30,40)	5.0	$\hat{\alpha}$	1.199	0.906	0.681	0.912	0.738	0.956
			$\hat{\lambda}$	1.097	0.934	1.124	0.956	0.972	0.976
			$\hat{R}(t)$	0.482	0.936	0.509	0.944	0.418	0.966
			$\hat{H}(t)$	0.387	0.894	0.330	0.908	0.359	0.956
		7.0	$\hat{\alpha}$	0.906	0.906	0.584	0.923	0.585	0.935
			$\hat{\lambda}$	1.042	0.935	1.064	0.954	1.064	0.964
			$\hat{R}(t)$	0.428	0.934	0.403	0.954	0.398	0.967
			$\hat{H}(t)$	0.297	0.907	0.274	0.924	0.288	0.946

Table 5. Failure times for a group of the Plane 720 observations..

1.2	**2.1**	**2.6**	**2.9**	**2.9**	**4.8**	**5.7**
5.9	**7.0**	**7.4**	**15.3**	**32.6**	**38.6**	**50.2**

The empirical Kolmogorov-Smirnov (KSD) statistic indicates a difference of 0.1637 between the empirical distribution and the cumulative distribution function (CDF) of the NP type distribution. Moreover, the p-value (PVKS) is computed as 0.8472, suggesting a good fit of the NP type model to the provided data. Hence, the NP type distribution is consistent with the supplied information. Fig.(3) depicts the theoretical and empirical probability density function (PDF), cumulative distribution function (CDF), and P-P plot of the NP type distribution using the dataset. The plot clearly

Table 3 continued..

n	(k,m)	τ	Par	MLE AL	MLE CP	Boot-p AL	Boot-p CP	Bayesian AL	Bayesian CP
80	(40, 50)	5.0	$\hat{\alpha}$	0.968	0.910	0.489	0.906	0.685	0.906
			$\hat{\lambda}$	0.998	0.927	0.904	0.906	0.922	0.906
			$\hat{R}(t)$	0.671	0.946	0.675	0.956	0.346	0.988
			$\hat{H}(t)$	0.379	0.896	0.232	0.924	0.299	0.946
		7.0	$\hat{\alpha}$	0.795	0.993	0.477	0.904	0.592	0.905
			$\hat{\lambda}$	0.889	0.904	0.998	0.906	0.865	0.907
			$\hat{R}(t)$	0.359	0.965	0.969	0.904	0.391	0.948
			$\hat{H}(t)$	0.292	0.918	0.273	0.964	0.299	0.955
	(40, 60)	5.0	$\hat{\alpha}$	0.868	0.900	0.481	0.916	0.615	0.926
			$\hat{\lambda}$	0.938	0.917	0.964	0.926	0.822	0.946
			$\hat{R}(t)$	0.371	0.943	0.375	0.954	0.342	0.968
			$\hat{H}(t)$	0.279	0.892	0.230	0.914	0.292	0.936
		7.0	$\hat{\alpha}$	0.775	0.913	0.447	0.924	0.532	0.935
			$\hat{\lambda}$	0.880	0.934	0.898	0.946	0.825	0.957
			$\hat{R}(t)$	0.350	0.945	0.369	0.964	0.331	0.978
			$\hat{H}(t)$	0.252	0.908	0.213	0.934	0.259	0.954

Table 4. continued.

n	(k,m)	τ	Par	MLE AL	MLE CP	Boot-p AL	Boot-p CP	Bayesian AL	Bayesian CP
80	(40, 50)	5.0	$\hat{\alpha}$	0.994	0.903	0.598	0.904	0.682	0.905
			$\hat{\lambda}$	1.162	0.903	1.134	0.905	0.983	0.907
			$\hat{R}(t)$	0.495	0.904	0.671	0.905	0.387	0.927
			$\hat{H}(t)$	0.395	0.902	0.292	0.904	0.667	0.905
		7.0	$\hat{\alpha}$	0.977	0.931	0.487	0.904	0.590	0.923
			$\hat{\lambda}$	0.967	0.907	0.949	0.906	0.905	0.938
			$\hat{R}(t)$	0.396	0.914	0.981	0.918	0.367	.986
			$\hat{H}(t)$	0.648	0.917	0.290	0.903	0.267	0.906
	(40, 60)	5.0	$\hat{\alpha}$	0.934	0.913	0.498	0.924	0.622	0.945
			$\hat{\lambda}$	1.062	0.943	1.104	0.955	0.923	0.967
			$\hat{R}(t)$	0.395	0.944	0.371	0.955	0.337	0.977
			$\hat{H}(t)$	0.295	0.912	0.232	0.924	0.267	0.945
		7.0	$\hat{\alpha}$	0.777	0.921	0.447	0.944	0.510	0.953
			$\hat{\lambda}$	0.917	0.947	0.939	0.956	0.915	0.978
			$\hat{R}(t)$	0.356	0.954	0.381	0.968	0.327	.976
			$\hat{H}(t)$	0.248	0.907	0.210	0.923	0.237	0.946

demonstrates that the NP type distribution fits the dataset very well.

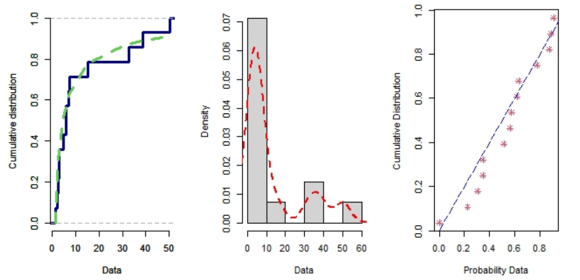


Figure 3. Data set's estimated CDF, pdf, and pp-plot..

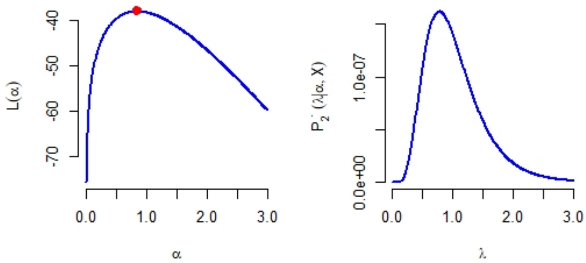


Figure 4. Plane 720 data: α profile MLE and λ full conditional distribution.

From Table (5), We analyze the data set under GHTCS and presumption $(n,(k,m),T) = (14,(8,12),7.5)$. Therefore, the Bayes estimates of $\alpha, \lambda, R(t)$, and $H(t)$ (at different times $t = 1.6$) together with their interval estimates are computed and shown in Table (6) for each generated sample. It is assumed that the priors of α and λ, i.e., $\nu = -1, c = 1, b = 0$, and $d \longrightarrow \infty$, are improper for computing the Bayes estimates and accompanying CIs because we without any prior knowledge of them. Following the MCMC methodology, we repeated the procedure a total of 11,000 times, discarding the first 1000 iterations as burn-in. The MCMC sampler was initialized with the MLEs of α and λ. As anticipated, Table (6) shows that the obtained estimates of α, λ, $R(t)$, and $H(t)$ exhibit similar performance, closely resembling each other. The behaviour of interval estimates for the unknown parameters exhibits a similar pattern. One of the key challenges when employing the MCMC procedure is ensuring the convergence of the Markov chains. To address this, we present trace plots (which provide a useful tool for assessing chain mixing) and density plots (which offer a smoothed histogram of outputs) in Figs. (5) and (6). In each plot, the dashed (- - -) line represents the Bayes estimate of α and λ, while the solid (-) lines represent the bounds of the confidence interval (CI) for the unknown quantities. These plots serve as visual evidence of the convergence and mixing of the MCMC chains.

Table 6. The estimation of the point and 95% interval.

Par.	MLE		Bootstrap		Bayesian	
	Point	ACI	Point	SBCI	Point	BCI
α	0.844	{0.191, 1.598}	1.105	{0.606, 1.972}	0.736	{0.367, 1.304}
λ	1.200	{0.150, 2.550}	1.426	{1.203, 2.138}	1.064	{0.206, 2.326}
$R(t)$	0.879	{0.567, 1.362}	0.935	{0.806, 1.188}	0.822	{0.379, 1.143}
$H(t)$	0.295	{0.149, 0.586}	0.367	{0.201, 0.635}	0.274	{0.124, 0.553}

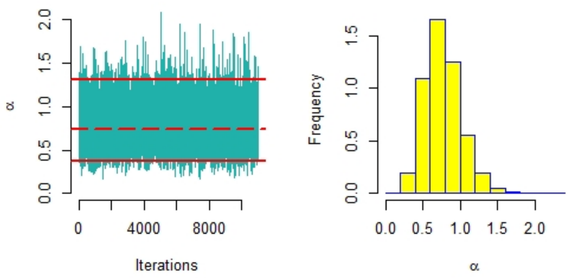


Figure 5. Plane 720 data's α trace and histogram plots (left and right, respectively).

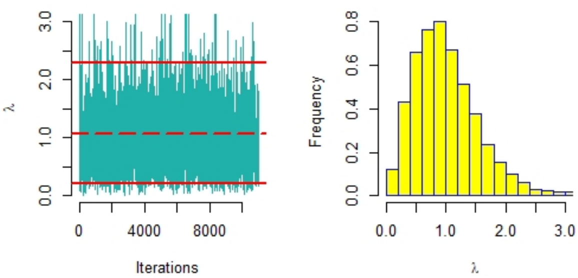


Figure 6. Plane 720 data's λ trace and histogram plots (left and right, respectively) .

6 Conclusions

This study introduces a new Pareto-type distribution suitable for modelling data exhibiting an increasing failure rate, which is relevant to various sectors encompassing disciplines such as medical science, engineering, chemistry, and other related fields. A GHTCS strategy is used to carry out a thorough statistical analysis, and methods such as maximum likelihood, bootstrapping, and Bayesian inferential approaches are used to estimate the unknown parameters of the new Pareto-type distribution. In addition, from the frequentest perspectives, approximate confidence intervals were generated for each unknown parameter. Based on the squared error loss function, Bayesian points and credible estimators are suggested using the outcome of the likelihood functions of the unknown parameters and bootstrapping. It should be noted that while the Bayesian estimators are not available in closed forms, they can be found via numerical integration, which necessitates the use of the Markov Chain Monte Carlo method. To assess the effectiveness of the suggested estimators, multiple numerical analyses were carried out, accounting for various effective sample sizes and censoring techniques. Monte Carlo simulations were utilized to analyse the results. The findings indicated that the bootstrap approach outperformed the likelihood approach in the classical viewpoint, while the Bayes-likelihood-based approach exhibited superior performance compared to the bootstrap approach in the Bayesian viewpoint, particularly in deriving point and/or interval estimates of the target parameter. Optimal censoring strategies were identified, and various optimal criteria were investigated. To evaluate the practical performance of the proposed estimators, an actual dataset obtained from the aeroplane engineering domain was examined. This real-world dataset offered a priceless chance to evaluate the estimators' performance in a pertinent application setting.

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