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A New Numerical Solution to the Lane- Emden Equations with Cylindrical and Planar Polytropes Using the Monte Carlo Method

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Abstract: Lane-Emden equations are important because they explain a star's internal structure, interstellar matter, molecular clouds, and even spiral arms, which can be computed and utilised to forecast different physical properties. The primary goal of this research is to develop a numerical solution for the Lane-Emden (LE) type equations using the Monte Carlo (MC) method. The MC method is increasingly becoming a valuable tool in various scientific fields and has many applications in multiple disciplines. This paper proposes an MC algorithm that can solve two key problems related to LE-type equations: a 1-D problem for a planar polytrope and a 2-D problem for a cylindrical polytrope. To gauge the performance and effectiveness of the numerical solutions generated by the proposed MC algorithm, we compared them with other numerical methods mentioned in previous literature. Our findings indicate that the MC solutions align with Runge-Kutta (RK) solutions and other numerical methods used to solve these problems.
Keywords: Lane-Emden equations; polytrope; Monte Carlo method.

1. Introduction

Over many years, the distributions of vast gas accumulations have been identified across various regions of the galaxy, including the interstellar medium, molecular clouds, and spiral arms. Immense, planar sheets and elongated cylindrical strands typically characterize these formations. Researchers often model these gas condensations as either unbounded, planar polytropes of finite thickness or infinite, cylindrical polytropes with a defined radius. Although these models reduce the dimensions of the LE equations from three to one or two, they provide useful insights into the behavior of self-gravitating gas distributions, which have become a central issue for many phenomena in astrophysics. [1] discussed the derivation of the 2-D LE equation model for self-gravitating gas distribution in the form of an unbounded, cylindrical polytrope with a finite radius and also focused on the 1-D LE equation model for an unbounded, planar polytrope with a finite thickness and its properties across the thickness.

The earliest discussions and analyses of the LE equations, including polytropic and isothermal varieties, were conducted by [2, 3]. The significance of the LE equation in various scientific disciplines has been questioned, and the reason for its widespread use lies in its ability to model a range of phenomena, such as radiative cooling, galactic cluster models, isothermal core analysis, convective stellar interiors, and the configuration of degenerate stars, as demonstrated by [4]. Many studies have employed both analytic and numerical approaches to solve these equations.

Several analytic solutions have been derived for the LE equations, including Runge-Kutta type methods (see [5]), Adomian decomposition methods ([6, 7]), Homotopy perturbation methods ([8, 9]), series expansion ([10]), accelerated power series method ([11]), variational iteration method ([12]), and differential transform method ([13]). Additionally, numerous computational approaches have been employed to provide numerical solutions to these equations, such as the Genetic Algorithm (e.g., [14]), Lattice Boltzmann method (e.g., [15]), Ant colony Algorithm (e.g., [16]), Artificial neural networks (e.g., [17, 18, 19, 20]).

Monte Carlo (MC) methods are a collection of statistical experiments that employ generated random numbers to approximate the expected value of functions based on a statistical model and provide solutions to complex integrals. These methods address two main classes of statistical problems: integration and estimation. By increasing samples of random numbers primarily in experimental studies, we can obtain a more accurate solution to the problem. The MC methods are a significant topic of investigation in various disciplines, including engineering, mathematics, and numerous scientific domains, such as physics, chemistry, biology, statistics, and artificial intelligence. An essential part of the MC integration is determining the region under a curve by counting the number of permitted random samples. According to deterministic analysis, the MC method demonstrates significant superiority. The experimenter can predict which scenario is more likely to occur and which outcomes will appear. Experimental processes have shown that more simulated random samples lead to more accurate solutions. When we analyze the MC findings, we can identify the required input values to achieve the specified results.

This technique was initially presented by [21], who devised a fresh and productive synthetic approach for acquiring numerical solutions to ordinary differential equations using the MC method. Later, [22] broadened this method to address various forms of ordinary differential equations. In addition, [23] refined this approach to solve more complex ordinary differential equations.

The authors of [24] have created a new algorithm that uses Monte Carlo (MC) methods to tackle four different types of problems in astrophysics. These problems include the positive and negative indices of the polytropic gas sphere, the isothermal gas sphere, and the white dwarf equation. In addition, [25] developed two computational algorithms based on MC methods to solve eleven LE-type equations, each with varying initial conditions. The present study makes a unique contribution by applying the MC method to obtain numerical solutions for key problems of the LE-type equations, including the 1-D LE problem for a planar polytrope and the 2-D LE problem for a cylindrical polytrope. The results of the MC findings will be compared with other available solutions to enhance the versatility and applicability of the investigated approach.

This paper starts with an introductory section and proceeds to the second section, which reviews the procedures for solving the two key problems related to the LE-type equations and proposes the implementation of the MC solver. The third section displays the performance of the MC findings and conducts a comparative study with other literature. Finally, the paper concludes in the fourth section.

2. Materials and methods

2.1 The Lane- Emden Equations

2.1.1 Implementation of the 1-D problem to Planar Polytrope

The design of the 1-D LE equation was based on considering a disc that contained a plane of radius a, height $r = \alpha t$, total thickness $2r$, surface area $dA = \pi a^2$, and total mass $M(r)$. To determine the polytropic LE equation, let us define the mass continuity and the hydrostatic equilibrium in the sense of a plane as

$$\frac{dm(r)}{dr} = 2\rho(r), \tag{1}$$

$$\frac{dP(r)}{dr} = -2\pi G\rho(r)m(r). \tag{2}$$

After integration with respect to r, Eq. (1) can be rewritten as

$$m(r) = \int_0^r 2\rho(r_1)\, dr_1. \tag{3}$$

Dividing $\rho(r)$ on both sides of Eq. (2), we can differentiate with respect to r and then combine the result with Eq. (2) to obtain

$$\frac{d}{dr}\left(\frac{1}{\rho(r)}\frac{dP(r)}{dr}\right) = -2\pi G\frac{dm(r)}{dr} = -4\pi G\rho(r). \tag{4}$$

We shall define the density and the pressure of a star to be

$$\rho(r) = \rho_c(r)y^n(r). \tag{5}$$

$$P(r) = {\rho_c}^{1+\frac{1}{n}}(r)y^{n+1}(r). \tag{6}$$

Inserting Eqs. (5) and (6) into Eq. (4), gets

$$\begin{aligned} &\frac{K}{\alpha^2}\frac{d}{dt}\left(\frac{1}{\rho_c(t)y^n(t)}\frac{d}{dt}\left[{\rho_c}^{1+\frac{1}{n}}(t)y^{n+1}(t)\right]\right) = -4\pi G\rho_c(t)y^n(t),\\ &\frac{1}{\alpha^2}\left(\frac{(n+1)K{\rho_c}^{\frac{1}{n}-1}(t)}{4\pi G}\right)\\ &\frac{d}{dt}\left(\frac{1}{y^n(t)}\frac{d}{dt}\left[y^{n+1}(t)\right]\right) = -y^n(t). \end{aligned} \tag{7}$$

Replacing $\frac{d}{dt}y^{n+1}(t) = (n+1)y^n(t)\frac{dy(t)}{dt}$ and

$\alpha = \left(\frac{(n+1)K{\rho_c}^{\frac{1}{n}-1}(t)}{4\pi G}\right)^{0.5}$, we get

$$\frac{d^2y(t)}{dt^2} = -y^n(t). \tag{8}$$

The Lane-Emden equation's shape is appropriate for a planar polytrope. However, the existence of a solution for equation (8) does not guarantee satisfaction, provided that the initial conditions $y'(0) = 0$ and $y(0) = 1$ are met. In light of the 1-D problem's implementation, we will assess some physical parameters for planar polytrope.

- **Mass of the planar polytropes**

Applying Eqs. (5) on Eq. (1) and integrate into t, we have

$$\begin{aligned} dm(r) &= 2\rho(r)dr = 2\rho_c(r)y^n(r)dr,\\ m(t) &= 2\alpha\rho_c(t)\int_0^t y^n(t_1)dt_1\\ &= 2\alpha\rho_c(t)\int_0^t\left(\frac{d}{dt_1}\left[-y'(t_1)\right]\right)dt_1\\ &= 2\alpha\rho_c(t)\left|-y'(t)\right|. \end{aligned} \tag{9}$$

The total mass (per unit area) of the plane is given as

$$M = 2\alpha\rho_c(t)\left|-y'(t)\right|_{t_1}.$$

Then the mass fraction is also computed as

$$\frac{m(t)}{M} = \frac{|y'(t)|}{|y'(t)|_{t_1}}.$$

- **Density ratio of the planar polytropes**

According to Eq. (5), the density ratio is given by

$$\frac{\rho(t)}{\rho_c(t)} = y^n(t). \tag{10}$$

- **Pressure ratio parameter to a planar polytropes**

Let the pressure denote as $P(t) = K\rho^\gamma(t)$, where $\gamma = 1 + \frac{1}{n}$ and $\rho(t) = \rho_c(t)y^n(t)$, then $P(t)$ takes the following form as

$$\begin{aligned} P(t) &= K{\rho_c}^{1+\frac{1}{n}}(t)\left(y^n(t)\right)^{1+\frac{1}{n}}\\ &= P_c(t)y^{n+1}(t), \end{aligned} \tag{11}$$

$$\frac{P(t)}{P_c(t)} = y^{n+1}(t).$$

Since the central pressure is given by

$$P_c(t) = K{\rho_c}^{1+\frac{1}{n}}(t).$$

2.1.2 The 2-D problem with Cylindrical Polytropes

One of the challenges we face is determining the variables needed to apply the 2-D LE equation to cylindrical polytropes. To address this, we must define a cylindrical disc with specific properties, including a radius of r, an axial length of dz, a mass of $m(r)$ and a surface area $dA = 2\pi r dz$. The equations for mass continuity and hydrostatic equilibrium within the context of the 2-D LE equation for cylindrical polytropes can be expressed as follows.

$$\frac{dm(r)}{dr} = 2\pi r\rho(r), \tag{12}$$

$$\frac{dP(r)}{dr} = \frac{-2G}{r}\rho(r)m(r). \tag{13}$$

For simplicity, multiple in $\frac{r}{\rho(r)}$ on both sides of Eq. (13) and differentiate with respect to r yields

$$\frac{d}{dr}\left(\frac{r}{\rho(r)}\frac{dP(r)}{dr}\right) = -2G\frac{dm(r)}{dr}. \tag{14}$$

Inserting Eq. (12) into Eq. (14) gives

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G\rho(r). \tag{15}$$

Set the density of the star defines in a dimensionless form with variables y and $r$as

$$\begin{gathered}\rho(r) = \rho_c(r)y^n(r),\\ r = \alpha t.\end{gathered} \tag{16}$$

The combination of Eqs. (16) with (11) holds the following formulae

$$\begin{aligned}P(r) &= {\rho_c}^{1+\frac{1}{n}}(r)y^{n+1}(r)\\ &= P_c(r)y^{n+1}(r).\end{aligned} \tag{17}$$

Applying Eqs. (16) and (17) on Eq. (13), gives

$$\frac{1}{\alpha^2}\left[\frac{(n+1)K}{4\pi G{\rho_c}^{1+\frac{1}{n}}}\right]\frac{1}{t}\frac{d}{dt}\left(t\frac{dy(t)}{dt}\right) = -y^n(t). \tag{18}$$

Let us regard α as a fixed value as

$$\alpha = \left[\frac{(n+1)K}{4\pi G{\rho_c}^{1+\frac{1}{n}}}\right]^{0.5}. \tag{19}$$

To shorten notation, we let Eq. (18) stand for

$$\frac{1}{t}\frac{d}{dt}\left(t\frac{dy(t)}{dt}\right) = y''(t) + \frac{y'(t)}{t} = -y^n(t). \tag{20}$$

This shape of the Lane- Emden equation is suitable for the cylindrical polytropes. With the initial condition, $y'(0) = 0$ and $y(0) = 1$, an analytical solution to equation (20) is impossible. Throughout this calculation, we want to measure some of the physical parameters for cylindrical polytropes.

- **Mass of the cylindrical polytrope**

Applying Eqs. (16) on Eq. (12) and integrate into t, we have

$$\begin{aligned}dm(r) &= 2\pi\rho r dr = 2\pi r\rho_c y^n r dr,\\ m(t) &= 2\pi\alpha^2\rho_c(t)\int_0^t y^n(t_1)\,t_1 dt_1\\ &= 2\pi\alpha^2\rho_c(t)\int_0^t\left(\frac{d}{dt_1}[-t_1 y'(t_1)]\right)dt_1\\ &= 2\pi\alpha^2\rho_c(t)\left|ty'(t)\right| \propto \left|ty'(t)\right|.\end{aligned} \tag{21}$$

The total mass (per unit area) of the disk is given as

$M = 2\pi\alpha^2\rho_c(t)\left|ty'(t)\right|_{t_1}$. Then the mass fraction is also computed as $\frac{m(t)}{M} = \frac{|ty'(t)|}{|ty'(t)|_{t_1}}$.

- **Density ratio parameter to a cylindrical polytrope**

The density ratio can be written as

$$\frac{\rho(t)}{\rho_c(t)} = y^n(t). \tag{22}$$

- **Pressure ratio parameter to a cylindrical polytrope**

Let the pressure denote as $P(t) = K\rho^\gamma(t)$, where $\gamma = 1 + \frac{1}{n}$ and $\rho(t) = \rho_c(t)y^n(t)$, then $P(t)$ takes the following form as

$$\begin{aligned}P(t) &= K{\rho_c}^{1+\frac{1}{n}}(t)\left(y^n(t)\right)^{1+\frac{1}{n}}\\ &= P_c(t)y^{n+1}(t),\\ \frac{P(t)}{P_c(t)} &= y^{n+1}(t).\end{aligned} \tag{23}$$

Since the central pressure equals $P_c(t) = K{\rho_c}^{1+\frac{1}{n}}(t)$

2.2 The MC Method

The validity of an explicit solution may not hold in numerous integration problems. To determine the area under the curve (integral), we can write the width of the interval $(b-a)$ multiplied by the mean value of the function $\langle f\rangle$ as

$$I = \int_a^b f(t)\ dt = (b-a)\langle f\rangle. \tag{24}$$

To calculate the mean value of a function f, we consider a group of variables t_i uniformly distributed over an interval $[a, b]$:

$$\langle f\rangle = \frac{1}{N}\sum_{i=1}^N f(t_i). \tag{25}$$

Then the approximate value of the integral I can be obtained in a convenient formula:

$$\int_a^b f(t)\,dt = \frac{(b-a)}{N}\sum_{i=1}^N f(t_i). \tag{26}$$

The MC integration of the function $f(t)$ over the interval $[a, b]$ can be performed using the following strategy:

Choose a value M such that $f(t) \le M$ is defined over the interval $[a, b]$, where M is the upper bound of $f(t)$.

Choose a value R such that $f(t) \geq R$is defined over the interval $[a,b]$, where is the lower bound of $f(t)$.

Simulate a group of variables t_i with Nvalues uniformly distributed from a to b

Simulate two groups of variables y, each with N values uniformly distributed from $[-R,0]$ and $[0,M]$, respectively.

Compute if $y \geq f(t)$, then $S \rightarrow$count positive random variables from $[0,M]$ greater than or equal $f(t)$, and equation (26) yields to

$$\int_a^b f(t)\ dt = M(b-a)\frac{S}{N}. \quad (27)$$

Compute if $y \leq f(t)$, then $S \rightarrow$ count negative random variables from $[-R,0]$ less than or equal $f(t)$, and equation (26) become

$$\int_a^b f(t)\ dt = R(b-a)\frac{S}{N}. \quad (28)$$

Previously, we proposed the general strategy of the MC integration method for functions $f(t)$ with explicit forms (27). When the function $f(t,y)$ has an implicit form, we can estimate the solution $y(t)$ using the following formulae:

$$\begin{aligned} y(t) &= y(t_0) + M\frac{S}{N}(b-a), \\ y(t) &= y(t_0) + R\frac{S}{N}(b-a). \end{aligned} \quad (29)$$

In the posterior steps, we provide a detailed description and discussion of the MC integration method employed in this study, as reported by [22].

The following strategy can be used to integrate the solutions of $y'(t)$ and $y(t)$:

First, to obtain the solution of $y'(t)$, we do the following steps:

Transform the ODEs into a convenient form $y''(t) = h(t,y,y')$.

Determine the values of M_1 and R_1 for the random variables generated, where $M_1$and R_1 represent the positive upper and negative lower boundaries of $h(t,y,y')$., respectively.

Divide the specified interval $[0,t_{final}]$ into m points with a step size $dt = 0.001$, where t_{final} is the final limit of the integration.

Simulate two groups, $[0,M_1]$ and $[R_1,0]$, each containing $N = 10^6$ random variables from the uniform distribution.

Let us focus on the random variables $JF_1 = h(t_i,y_i,y_i')$, with $i = 0,1,\dots,m$.

Begin with initial conditions at $t \rightarrow t_0$, $y \rightarrow y_0$ and $y' \rightarrow y_0'$.

If $JF_1 \geq 0$, the following actions should be taken:

$S_1 \rightarrow$Calculate the number of random variables $[0,M_1]$ that are equal to or less than JF_1.

Determine $y_{i+1}'(t) = y_i'(t) + M_1 \times \frac{S_1}{N}dt$.

Conversely, if $JF_1 \leq 0$perform the following tasks:

$S_1 \rightarrow$ Count the number of random variables for the interval $[R_1,0]$, which is greater than or equal to JF_1.

Compute $y_{i+1}'(t) = y_i'(t) + R_1 \times \frac{S_1}{N}dt$.

To obtain the solution $y(t)$, follow these steps:

Classify $M_2$and R_2 as the positive upper and negative lower boundaries of the solution for $y'(t)$.

Create two groups of random variables as $[R_2,0]$ and $[0,M_2]$.

Define $JF_2 = y_{i+1}'(t)$.

If $JF_2 \geq 0$, then complete the following:

$S_2 \rightarrow$ Calculate the number of random variables from $[0,M_2]$, which is equal to or less than JF_2.

Compute $y_{i+1}(t) = y_i(t) + M_2 \times \frac{S_2}{N}dt$.

On the other hand, if $JF_2 \leq 0$, perform these actions:

$S_2 \rightarrow$ Calculate the number of random variables for the interval $[R_2,0]$, which is greater than or equal to JF_2.

Compute $y_{i+1}(t) = y_i(t) + R_2 \times \frac{S_2}{N}dt$.

Execute the second iteration with $t_{i+1} = t_i + \Delta t$.

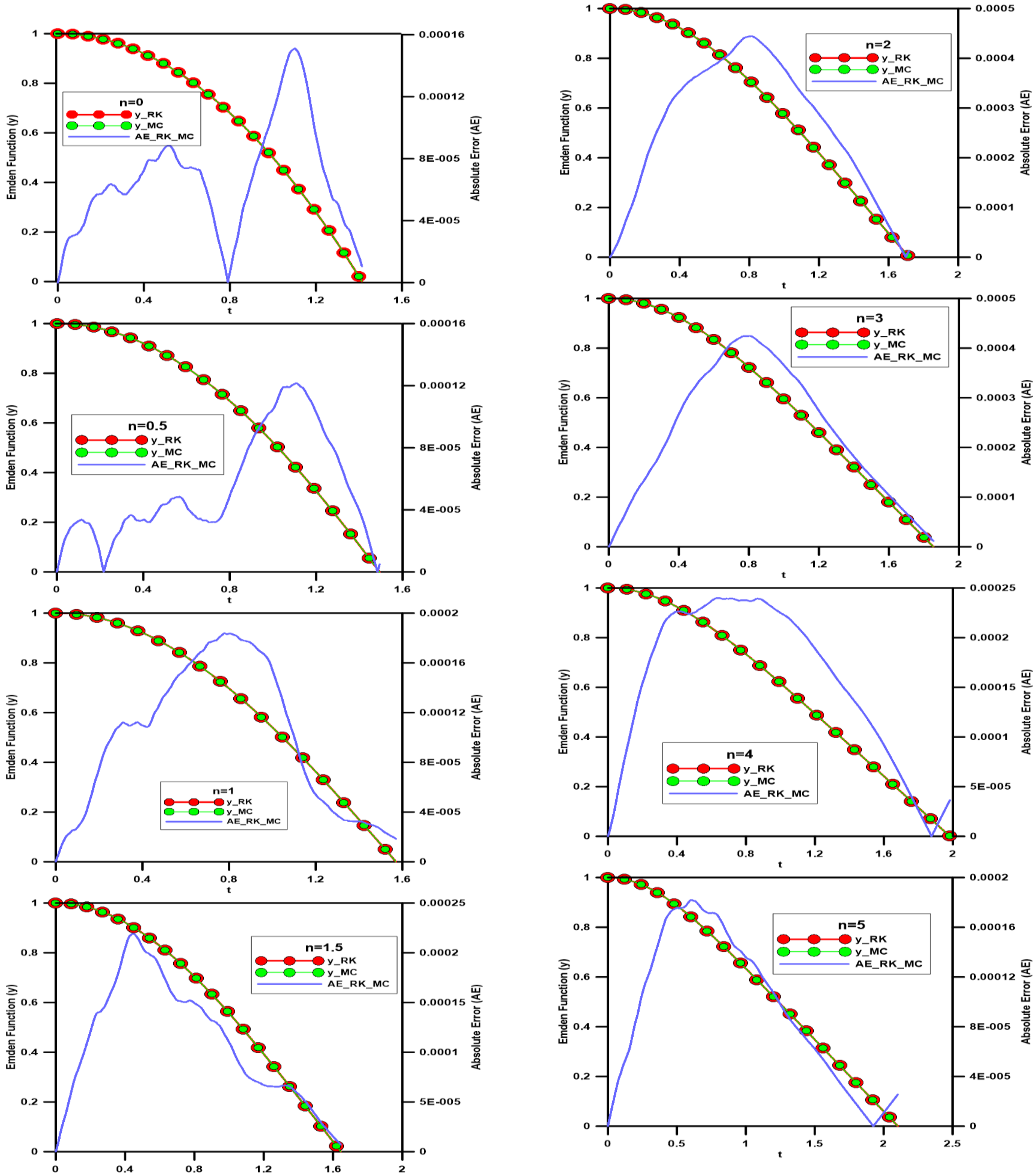
Repeat the steps mentioned above until $t_{i+1} = t_{final}$.

3. Results and Discussion:

This section provides the numerical outcomes and explores the consequences of the findings from the MC method on the other techniques employed in this study. The software package R is used to develop the computational framework procedure equipped with solver codes. The RK technique solver is performed to the ordinary differential equations based on the R package function rkMethod. The computation of MC involves the utilization of a million random samples. Throughout the posterior subsections, we observe the numerical findings for each problem in more detail.

3.1 Planar polytrope

Planar polytrope uses a 1-D LE equation with polytrope indices $n = 0,0.5,1,1.5,2,3,4,5,6,10$, which is examined to showcase the effectiveness of the MC method. The results of the MC analysis with polytrope indices $n = 0,0.5,1,1.5,2,3,4,5,6,10$ are organized in Tables **1-10** as well as depicted in Fig. **1**. To assess the accuracy of the MC findings, the absolute error can be calculated using the following formula $AE = |y_{MC}(t) - y_{RK}(t)|$, where $y_{MC}(t)$ and $y_{RK}(t)$are the numerical solutions obtained from the MC and RK methods. Moreover, tables **1-10** and Figure **4** depict some of the principal physical attributes, including mass, density ratio, and pressure ratio, which have been measured. For the sake of comparison, we utilized the MC method to calculate the Emden radius (t) and the first derivative of the Emden function $y'(t)$. We then compared these results to those obtained through the RK method ([5]) and the Euler-Richardson method ([1]). Our findings, displayed in Table **11**, show that the results for various polytropic indices exhibit good agreement.



n=0
y_RK
y_MC
AE_RK_MC
n=0.5
n=1
n=1.5
n=2
n=3
n=4
n=5
Emden Function (y)
Absolute Error (AE)
t

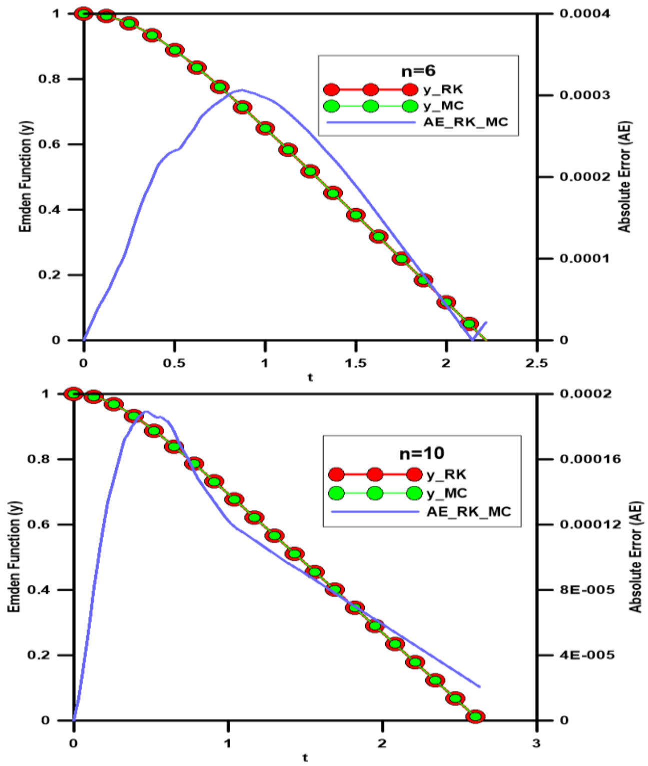


Figure 1: Performance of the RK and MC solutions for the plane with polytrope indexes n=0, 0.5, 1, 1.5,2,3,4,5,6 and 10.

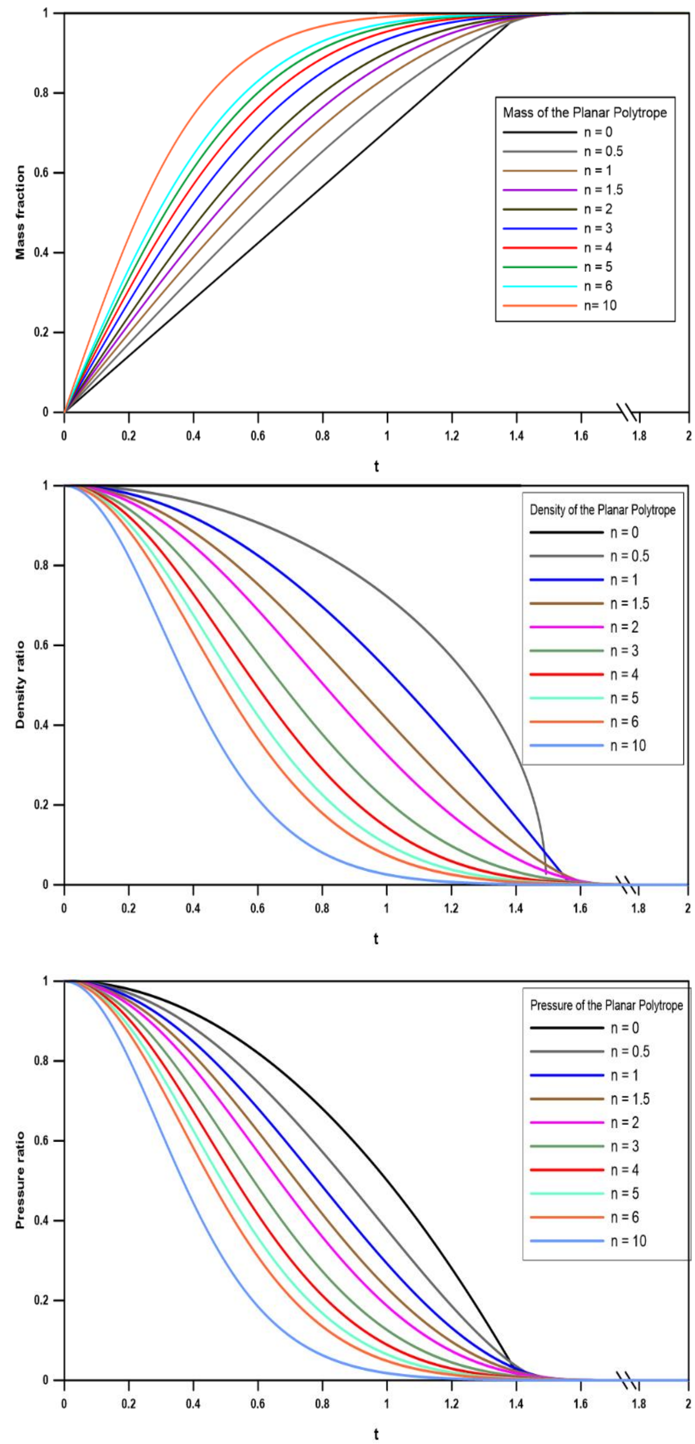
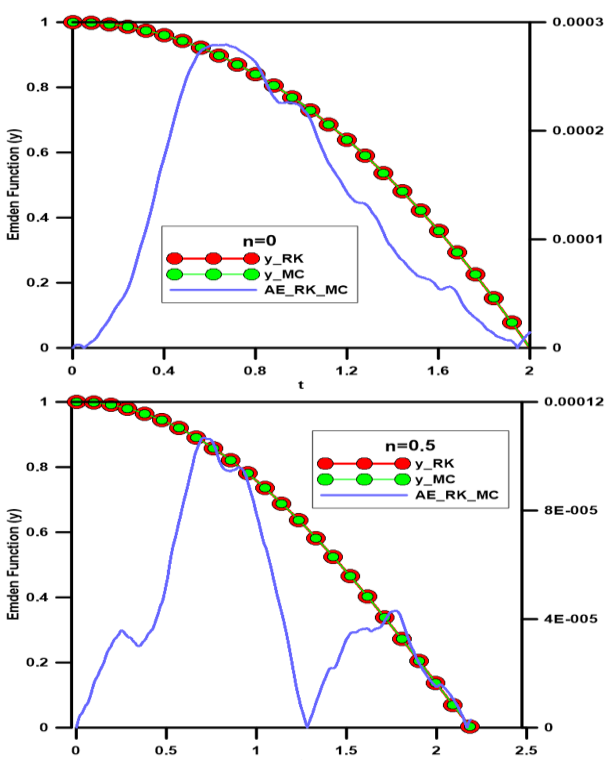
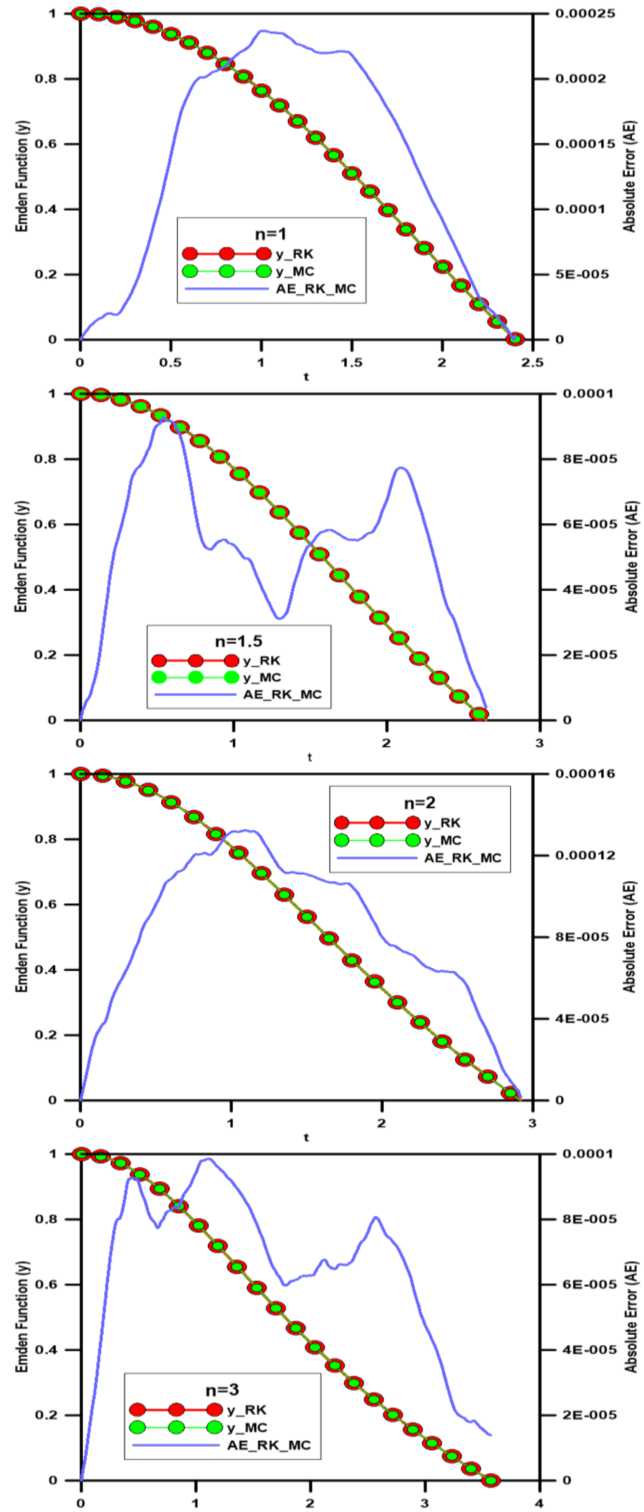


Figure 2: Plots of Mass fraction, Density ratio, and Pressure ratio parameters for a plane with different values of Polytrope indexes

3.1 Cylindrical polytropes

For cylindrical polytropes, a 2-D LE equation using polytrope indices $n = 0,0.5,1,1.5,2,3,4,5,6,10$ is employed to demonstrate the effectiveness of the MC method. The outcomes of the MC findings are presented in Tables 12-21. Furthermore, the results for the polytrope indices $n = 0,0.5,1,1.5,2,3,4,5,6,10$ are presented in Fig. 3, where the MC and RK solutions are displayed in various colors to ensure that they overlap and cannot be distinguished. For comparison, we computed the absolute error between the MC and RK solutions. According to the findings, the results from the two methods, MC and RK, are in good agreement, as depicted in Tables 12-21 and Fig. 3. The mass, density ratio, and pressure ratio parameters were calculated for different values of the polytropic indices and are presented in the same tables, as well as plotted in Fig. 4. The outcome of the MC method can also be inferred by examining the Emden radius (t) and the first derivative of the Emden function $y'(t)$, which was obtained using the MC method, in comparison to those obtained through the RK method ([5]) and the Euler-Richardson method ([1]). The results for various polytropic indices are presented in Table 22.





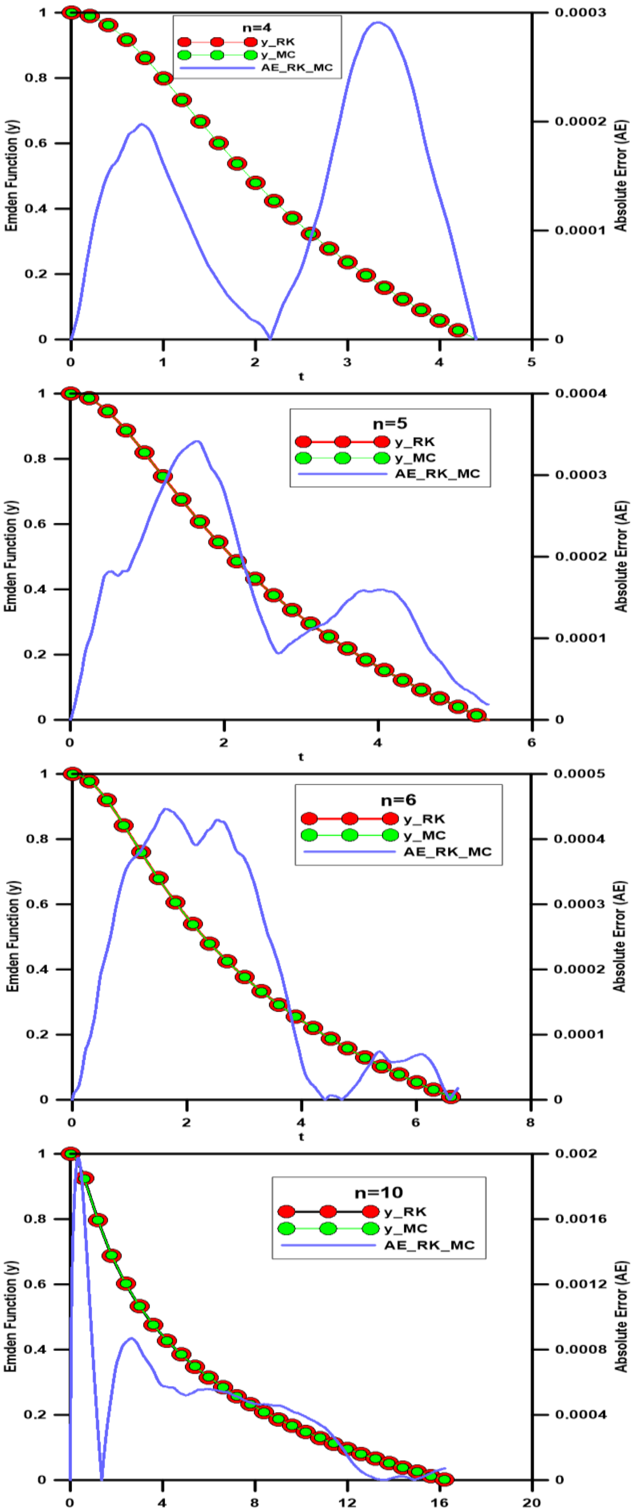


Figure 3: Comparison between the RK and MC solutions for the cylinder with polytrope indexes n=0, 0.5, 1, 1.5, 2, 3, 4, 5, 6, and 10.

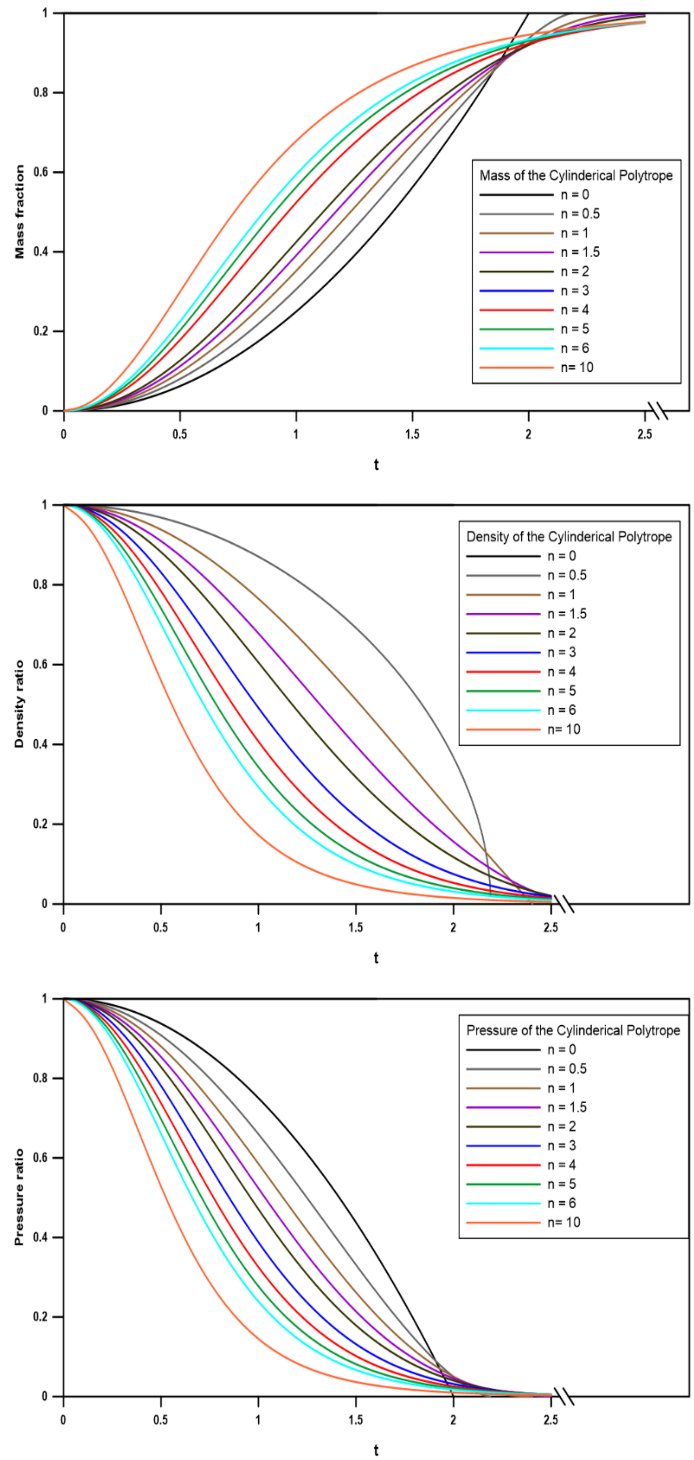


Figure 4: Plots of Mass fraction, Density ratio, and Pressure ratio parameters for a cylinder with different values of Polytrope indexes

4. Conclusion

The current study used the MC solver to obtain numerical solutions for two key LE-type equation problems. The solver successfully provided accurate solutions for a 1-D planar polytrope and a 2-D cylindrical polytrope. The results obtained using the MC and RK methods are presented in

Tables **1-10** and **12- 21** for each problem, along with polytropic indices $n = 0, 0.5, 1, 1.5, 2, 3, 4, 5, 6, 10$, and are also shown in Figures **1** and **3**. Additionally, we calculated the absolute error between the MC and RK solutions, which showed good agreement. Furthermore, we evaluated some key physical attributes, such as mass, density ratio, and pressure ratio. A comparison was conducted between the Emden radius and the first derivative of the Emden function using the MC method, which was subsequently compared to other relevant studies and presented in Tables **11** and **22**. The results demonstrated that the MC method is a potent tool and a suitable solver for various differential equation scenarios. According to the nonlinearity of the LE equation, which has a singularity at t=0, the MC can overcome this drawback by initializing the iteration process with t=0.001 to obtain the next iterative solution. The posterior initial conditions can then be used as the previous stage of the numerical solutions to generate the desired result.

In Tables **11** and **22**, we compared the zeroth of the Emden functions and their first derivative of the MC solution with two different numerical solutions; the maximum absolute error is 0.001, which reflects good agreement. The precision in the Emden function demonstrates the accuracy of determining the density and pressure profiles as indicated by equations (16) and (17). In contrast, the precision in the calculation of the first derivative of the Emden function reflects the accuracy of determining the mass of the polytrope as followed from equation (21).
In Figures **3** and **4**, we plotted some physical quantities of the polytropes, like mass, density, and pressure. The mass of the polytrope indicates some interesting behaviors for plane and cylindrical polytropes. For all ranges of the polytropic indexes, the mass increases with increasing the distance (t) for about 80% of the total distance inside the polytrope; after that, the mass tends to be constant instead of increasing the distance (t). The density profiles for the polytropic plane and cylinder behave similarly; parameter magnitudes for the same polytropic indices fall rapidly from cylindrical to planar models. As the polytropic index increases, the density profile approaches the polytropic planar core plane. The pressure profile changes dramatically for each polytropic shape. The planar polytrope has a little change with raising the polytropic index. In contrast, the cylindrical polytropic has a large change (this change may not be observed clearly because of the large value of the distance *t* taken by the polytrope with n>5); these behaviors are similar to the analytical model results of the N-dimensional polytrope obtained by [**26**].

CRediT authorship contribution statement:

Conceptualization, M.N. and S.E.; methodology, S.E.; software, S.E.; validation, S.E., H.A. and A.S.; formal analysis, M.N.; investigation, S.E., G.A.; writing—original draft preparation, S.E., M.N,.; writing—review and editing, S.E, M.N., H.A. All authors have read and agreed to the published version of the manuscript.

Data availability statement

The data used to support the findings of this study are available from the corresponding author upon request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Table 1: RK and MC solutions of Emden Function of a plane with polytrope index n= 0 and some physical parameters.

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9950000	0.9949673	3.269890e-05	-0.100	0.07077141	1	0.9949673
0.2	0.9800000	0.9799429	5.712621e-05	-0.200	0.1415428	1	0.9799429
0.3	0.9550000	0.9549426	5.738149e-05	-0.300	0.2123142	1	0.9549426
0.4	0.9200000	0.9199277	7.233702e-05	-0.400	0.2830856	1	0.9199277
0.5	0.8750000	0.8749127	8.734375e-05	-0.500	0.3538570	1	0.8749127
0.6	0.8200000	0.8199263	7.371588e-05	-0.600	0.4246285	1	0.8199263
0.7	0.7550000	0.7549443	5.567380e-05	-0.700	0.4953999	1	0.7549443
0.8	0.6800000	0.6800064	6.448010e-06	-0.800	0.5661713	1	0.6800064
0.9	0.5950000	0.5950657	6.569203e-05	-0.900	0.6369427	1	0.5950657
1	0.5000000	0.5001093	1.092898e-04	-1.000	0.7077141	1	0.5001093
1.1	0.3950000	0.3951510	1.509897e-04	-1.100	0.7784855	1	0.3951510
1.2	0.2800000	0.2800950	9.499326e-05	-1.200	0.8492569	1	0.2800950
1.3	0.1550000	0.1550511	5.115970e-05	-1.300	0.9200283	1	0.1550511
1.414	0.0003020	0.0003124	1.047774e-05	-1.414	1.0007077	1	0.0003124

Table 2: RK and MC solutions of Emden Function of a plane with polytrope index n= 0.5 and some physical parameters.

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9950021	0.9949694	3.270843e-05	-0.09991601	0.0864972	0.9974815	0.9924636
0.2	0.9800334	0.9800221	1.124680e-05	-0.19933922	0.1725679	0.9899607	0.9701834
0.3	0.9551693	0.9551979	2.861916e-05	-0.2977532	0.2577648	0.9773423	0.9335553
0.4	0.9205362	0.9205698	3.355206e-05	-0.3946216	0.3416237	0.9594633	0.8832529
0.5	0.8763133	0.8763568	4.352535e-05	-0.4894439	0.4237114	0.9361393	0.8203921
0.6	0.8227340	0.8227765	4.249631e-05	-0.5816594	0.5035422	0.9070703	0.7463161
0.7	0.7600893	0.7601216	3.233005e-05	-0.6706769	0.5806047	0.8718495	0.6627117
0.8	0.6894873	0.6895358	4.852291e-05	-0.7550148	0.6543348	0.8299282	0.5725787
0.9	0.6090817	0.6091650	8.321424e-05	-0.8364558	0.7241193	0.7804902	0.4754473
1	0.5216345	0.5217391	1.046331e-04	-0.9116928	0.7892520	0.7223151	0.3768600
1.1	0.4269736	0.4270947	1.210819e-04	-0.9806004	0.8489053	0.6535248	0.2791170
1.2	0.3257898	0.3258924	1.026223e-04	-1.0419907	0.9015558	0.5717817	0.1860422
1.3	0.2200085	0.2200828	7.429647e-05	-1.093726	0.9472443	0.4679623	0.1032474
1.4	0.1085243	0.1085642	3.988648e-05	-1.134159	0.9821261	0.3277649	0.0357709
1.493	0.0007717	0.0007673	4.426231e-06	-1.155135	1.0000000	0.0277014	0.0000212

Table 3: RK and MC solutions of Emden Function of a plane with polytrope index n= 1 and some physical parameters.

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9950042	0.9949765	2.762428e-05	-0.09983888	0.09985088	0.9949765	0.9899783
0.2	0.9800666	0.9799918	7.480784e-05	-0.19867266	0.1986966	0.9799918	0.9603839
0.3	0.9553365	0.9552262	1.102561e-04	-0.2955089	0.2955445	0.9552262	0.9124572
0.4	0.9210610	0.9209513	0.0001097240	-0.3893792	0.3894261	0.9209513	0.8481512
0.5	0.8775826	0.8774492	0.0001333289	-0.4793325	0.4793901	0.8774492	0.7699172
0.6	0.8253356	0.8251801	0.0001555189	-0.5644799	0.5645478	0.8251801	0.6809222
0.7	0.7648422	0.7646673	0.0001748893	-0.6440035	0.6440809	0.7646673	0.5847161
0.8	0.6967067	0.6965232	0.0001835113	-0.7170861	0.7171724	0.6965232	0.4851446
0.9	0.6216100	0.6214338	0.0001762133	-0.7830291	0.7831233	0.6214338	0.3861799
1	0.5403023	0.5401480	1.543029e-04	-0.8411594	0.8412605	0.5401480	0.2917599
1.1	0.4535961	0.4534952	1.009304e-04	-0.8908991	0.8910062	0.4534952	0.2056579
1.2	0.3623578	0.3623037	5.403848e-05	-0.9317543	0.9318664	0.3623037	0.13126398
1.3	0.2674988	0.2674616	3.727662e-05	-0.9633086	0.9634244	0.2674616	0.07153568
1.4	0.1699671	0.1699346	3.246090e-05	-0.9852452	0.9853636	0.1699346	0.028877796
1.5	0.0707372	0.0707110	2.617467e-05	-0.9973455	0.9974654	0.0707110	5.000049e-03
1.571	-0.0002036	-0.0002221	1.844080e-05	-0.9998805	1.0000008	-0.0002221	4.933463e-08

Table 4: RK and MC solutions of Emden Function of a plane with polytrope index n= 1.5 and some physical parameters.

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9950062	0.9949386	6.762384e-05	-0.099746582	0.11154201	0.9924175	0.9873945
0.2	0.9800996	0.9799790	1.206092e-04	-0.19799973	0.2214140	0.9701193	0.9506965
0.3	0.9555017	0.9553490	0.0001527413	-0.2933082	0.3279931	0.9337768	0.8920827
0.4	0.9215747	0.9213704	0.0002043214	-0.3843234	0.4297711	0.8844054	0.8148649
0.5	0.8788103	0.8786046	0.0002056655	-0.4698072	0.5253638	0.8235505	0.7235753
0.6	0.8278157	0.8276299	0.0001857624	-0.5487101	0.6135973	0.7529295	0.6231470
0.7	0.7692958	0.7691436	0.0001521666	-0.6201474	0.6934823	0.6745453	0.5188222
0.8	0.7040330	0.7038836	0.0001493691	-0.6834454	0.7642656	0.5905427	0.4156733
0.9	0.6328657	0.6327338	0.0001318393	-0.7381816	0.8254745	0.5033054	0.3184583
1	0.5566644	0.5565544	1.099812e-04	-0.7841167	0.8768416	0.4152040	0.2310836
1.1	0.4763086	0.4762316	7.703884e-05	-0.8212849	0.9184052	0.3286452	0.15651123
1.2	0.3926623	0.3925953	6.700660e-05	-0.8499376	0.9504461	0.2459901	0.09657457
1.3	0.3065502	0.3064842	6.594184e-05	-0.8706471	0.9736046	0.1696728	0.05200204
1.4	0.2187334	0.2186752	5.819582e-05	-0.8841826	0.9887408	0.10225846	0.022361389
1.5	0.129883696	0.129846926	3.677015e-05	-0.8915316	0.9969588	0.04678940	6.07546e-03
1.6	0.040552168	0.0405368702	1.529762e-05	-0.8941007	0.9998317	8.16160e-03	3.30845e-04
1.645	0.000304863	0.0002977462	7.116731e-06	-0.8942512	1.0000000	5.13770e-06	1.52973e-09

Table 5: RK and MC solutions of Emden Function of a plane with polytrope index n= 2 and some physical parameters.

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9950083	0.9949268	8.155087e-05	-0.09967332	0.12211783	0.9899622	0.9848574
0.2	0.9801324	0.9799453	0.0001871190	-0.1973647	0.2418317	0.9604172	0.9410345
0.3	0.9556650	0.9553900	0.0002749910	-0.2911825	0.3567968	0.9129185	0.8720514
0.4	0.9220777	0.9217442	0.0003334671	-0.3794168	0.4650044	0.8497391	0.7831254
0.5	0.8799988	0.8796345	0.0003642757	-0.4606701	0.5646286	0.7738485	0.6806233
0.6	0.8301837	0.8297969	0.0003868171	-0.5338519	0.6543042	0.6885810	0.5713674
0.7	0.7734814	0.7730674	0.000414000	-0.5981894	0.7331123	0.5976338	0.4620107
0.8	0.7107981	0.7103543	0.0004437775	-0.6533033	0.8005834	0.5046425	0.3584471
0.9	0.6430613	0.6426403	0.0004209742	-0.6991483	0.8567311	0.4130390	0.2654018
1	0.5711855	0.5708062	0.0003793144	-0.7359977	0.9019380	0.3258719	0.1859799
1.1	0.4960413	0.4957113	0.0003299476	-0.7644780	0.9369055	0.2457772	0.12181099
1.2	0.4184288	0.4181409	0.0002878241	-0.7854051	0.9626285	0.1748697	0.07310853
1.3	0.3390564	0.3388165	0.0002398522	-0.7997997	0.9802764	0.11480836	0.03889499
1.4	0.2585244	0.2583343	1.900122e-04	-0.8087776	0.9912963	0.06673709	0.017240364
1.5	0.17731338	0.17718560	1.277808e-04	-0.8135804	0.9971641	0.03167132	0.0055626951
1.6	0.095777053	0.095712981	6.407240e-05	-0.8155024	0.9995611	0.009147349	8.768242e-04
1.717	0.0002574759	0.0002697858	1.230994e-05	-0.8158610	1.0000000	2.361578e-08	1.963619e-11

Table 6: RK and MC solutions of Emden Function of a plane with polytrope index n= 3 and some physical parameters.

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9950125	0.9949415	7.091727e-05	-0.099500559	0.140789176	0.9849013	0.9799192
0.2	0.9801976	0.9800650	1.326534e-04	-0.1960600	0.2774168	0.9413792	0.9226128
0.3	0.9559859	0.9557943	0.0001915167	-0.2869974	0.4060895	0.8731591	0.8345605
0.4	0.9230532	0.9227865	0.0002667504	-0.3700917	0.5236645	0.7857849	0.7251117
0.5	0.8822664	0.8819411	0.0003252552	-0.4437555	0.6278957	0.6859916	0.6050042
0.6	0.8346167	0.8342534	0.0003633237	-0.5071005	0.7175263	0.5806226	0.4843864
0.7	0.7811526	0.7807430	0.0004096645	-0.5598887	0.7922194	0.4759094	0.3715629
0.8	0.7229170	0.7224927	0.0004243192	-0.6025014	0.8525146	0.3771381	0.2724795
0.9	0.6608952	0.6604940	0.0004012600	-0.6356992	0.8994880	0.2881420	0.1903161
1	0.5959766	0.5956109	0.0003656786	-0.6606097	0.9347354	0.2112944	0.12584922
1.1	0.5289300	0.5286034	0.0003265130	-0.6784632	0.9599974	0.1477032	0.07807643
1.2	0.4603925	0.4601186	0.0002738976	-0.6906231	0.9772031	0.09741128	0.04482074
1.3	0.3908686	0.3906431	0.0002255700	-0.6984019	0.9882098	0.05961291	0.023287371
1.4	0.3207383	0.3205570	0.0001813868	-0.7029695	0.9946728	0.03293940	0.010558952
1.5	0.2502703	0.2501290	1.412444e-04	-0.7053510	0.9980426	0.015649203	0.0039143197
1.6	0.17963921	0.17953511	1.040978e-04	-0.7063708	0.9994854	0.005786930	1.038957e-03
1.7	0.10894571	0.10887852	6.719905e-05	-0.7066853	0.9999305	0.0012907037	1.405299e-04
1.854	5.280483e-05	4.139102e-05	1.141380e-05	-0.7067344	1.0000000	7.091180e-14	2.935112e-18

Table 7: RK and MC solutions of Emden Function of a plane with polytrope index n= 4 and some physical parameters.

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9950166	0.994936	8.062674e-05	-0.09933347	0.15701168	0.9800178	0.9749350
0.2	0.9802621	0.9801082	0.0001539140	-0.1947724	0.3079293	0.9229510	0.9044200
0.3	0.9562994	0.9560949	0.0002044197	-0.2828643	0.4473735	0.8357420	0.7989233
0.4	0.9239904	0.9237643	0.0002261094	-0.3611582	0.5712810	0.7282126	0.6726758
0.5	0.8844004	0.8841752	0.0002251556	-0.4281923	0.6773097	0.6110815	0.5403707
0.6	0.8386924	0.8384554	0.0002370749	-0.4834581	0.7647713	0.4941114	0.4143809
0.7	0.7880300	0.7877909	0.0002390811	-0.5273986	0.8342216	0.3850465	0.3034275
0.8	0.7335004	0.7332627	0.0002376581	-0.5610255	0.8874125	0.2889785	0.2119815
0.9	0.6760630	0.6758257	0.0002372954	-0.5858106	0.9266409	0.2084929	0.14098516
1	0.6165230	0.6162981	0.0002248947	-0.6033456	0.9544169	0.1441330	0.08891072
1.1	0.5555256	0.5553150	0.0002105547	-0.6152141	0.9732032	0.09496401	0.05280767
1.2	0.4935646	0.4933755	0.0001890981	-0.6228512	0.9852794	0.05913199	0.02923398
1.3	0.4310010	0.4308355	0.0001654748	-0.6274574	0.9925685	0.03435545	0.014844221
1.4	0.3680850	0.3679424	0.0001426594	-0.6300262	0.9966390	0.01825550	0.006743714
1.5	0.3049802	0.3048606	1.195496e-04	-0.6313163	0.9986959	0.008590939	0.0026333394
1.6	0.2417851	0.241693	9.185897e-05	-0.6318664	0.9995922	0.003386123	0.0008247507
1.7	0.1785535	0.17849362	5.990142e-05	-0.6320567	0.9999100	0.0010034601	1.811813e-04
1.8	0.11531047	0.1152842	2.619737e-05	-0.6321038	0.9999901	1.733002e-04	2.036340e-05
1.9	0.052065112	0.052073175	8.062707e-06	-0.632109	0.9999998	7.029223e-06	3.828867e-07
1.982	0.0002037603	0.0002399847	3.622439e-05	-0.6321093	1.0000000	1.892132e-14	7.960088e-19

Table 8: RK and MC solutions of Emden Function of a plane with polytrope index n= 5 and some physical parameters.

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9950207	0.9949755	4.517393e-05	-0.099166761	0.17176616	0.9751095	0.9702294
0.2	0.9803260	0.9802421	8.388820e-05	-0.1935361	0.3352485	0.9049878	0.8871560
0.3	0.9566058	0.9564826	0.000123191	-0.2790592	0.4834787	0.8003990	0.7657067
0.4	0.9248916	0.9247316	0.0001599863	-0.3530285	0.6116742	0.6759843	0.6253085
0.5	0.8864138	0.8862384	0.0001754629	-0.4141677	0.7176292	0.5464399	0.4845106
0.6	0.8424578	0.8422761	0.0001817502	-0.4625932	0.8016252	0.4236403	0.3570482
0.7	0.7942459	0.794072	0.0001730155	-0.4994794	0.8655121	0.3154608	0.2507045
0.8	0.7428577	0.742688	0.0001688781	-0.5264488	0.9122105	0.2257455	0.16781913
0.9	0.6891914	0.6890437	0.0001476596	-0.5453649	0.9450017	0.1551232	0.10702390
1	0.6339554	0.6338196	0.0001358642	-0.5581011	0.9670979	0.10212864	0.06483267
1.1	0.5776830	0.5775590	0.0001239716	-0.5663231	0.9813581	0.06414639	0.03711744
1.2	0.5207571	0.5206497	1.073875e-04	-0.5713792	0.9900902	0.03816938	0.019919288
1.3	0.4634407	0.463351	8.920380e-05	-0.5743218	0.9950902	0.02129585	0.009896083
1.4	0.4059050	0.4058297	7.535966e-05	-0.5759122	0.9977702	0.010969466	0.004467482
1.5	0.3482555	0.3481933	6.224183e-05	-0.5766936	0.9990919	0.005095931	0.0017820596
1.6	0.2905524	0.2905046	4.783195e-05	-0.5770366	0.9996841	0.0020577963	0.0006010600

1.7	0.2328272	0.2327949	3.232501e-05	-0.5771651	0.9999122	0.0006788382	1.591625e-04
1.8	0.1750945	0.1750765	1.794488e-05	-0.5772011	0.9999841	1.628643e-04	2.879835e-05
1.9	0.11735979	0.11735619	3.602126e-06	-0.5772092	0.9999988	2.191890e-05	2.612378e-06
2.0	0.0596247854	0.0596354871	1.070176e-05	-0.5772096	1.0000000	7.307557e-07	4.498095e-08
2.103	0.0001577078	0.0001831541	2.544623e-05	-0.5772096	1.0000000	-3.899254e-19	3.774848e-23

Table 9: RK and MC solutions of Emden Function of a plane with polytrope index n= 6 and some physical parameters.

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9950248	0.9949770	4.786144e-05	-0.09901196	0.18533068	0.9702377	0.9653642
0.2	0.9803891	0.9802962	9.290702e-05	-0.1923126	0.3599710	0.8874501	0.8699640
0.3	0.956905	0.9567491	1.563680e-04	-0.2753271	0.5153574	0.7669874	0.7338145
0.4	0.925759	0.9255494	0.0002098654	-0.3451967	0.6461395	0.6286337	0.5818316
0.5	0.8883179	0.8880857	0.0002321835	-0.4011369	0.7508483	0.4906020	0.4356967
0.6	0.845951	0.8456948	0.0002567647	-0.4438366	0.8307737	0.3658321	0.3093823
0.7	0.7999030	0.7996184	0.0002845503	-0.4750254	0.8891531	0.2613947	0.2090160
0.8	0.7512162	0.7509150	0.0003011463	-0.4968996	0.9300971	0.1792853	0.13462806
0.9	0.7007147	0.7004101	0.0003046507	-0.5116179	0.9576468	0.11806313	0.08269261
1	0.6490148	0.6487178	0.0002969736	-0.5211258	0.9754438	0.07453065	0.04834936
1.1	0.5965558	0.5962722	0.0002836758	-0.5270133	0.9864639	0.04494353	0.026798577
1.2	0.5436375	0.5433739	0.0002635890	-0.5304919	0.9929752	0.02573907	0.013985940
1.3	0.4904550	0.4902144	0.0002405764	-0.5324293	0.9966016	0.013877664	0.006803031
1.4	0.437129	0.4369129	0.0002162700	-0.5334401	0.9984936	0.006956150	0.0030392316
1.5	0.3837309	0.383541	0.0001891440	-0.5339278	0.9994066	0.003183288	0.0012209240
1.6	0.3302992	0.3301389	0.0001602777	-0.5341372	0.9997985	0.0012947335	4.274419e-04
1.7	0.2768537	0.2767229	1.307492e-04	-0.5342133	0.9999409	0.0004490251	1.242555e-04
1.8	0.2234032	0.2233018	1.013395e-04	-0.5342384	0.9999879	1.239806e-04	2.768509e-05
1.9	0.1699513	0.1698794	7.186011e-05	-0.5342442	0.9999988	2.403502e-05	4.083054e-06
2.0	0.11649906	0.11645681	4.224726e-05	-0.534244	1.0000000	2.494534e-06	2.905055e-07
2.1	0.063046815	0.063034176	1.263844e-05	-0.5342449	1	6.272728e-08	3.953963e-09
2.217	0.000507684	0.0005296903	2.200633e-05	-0.5342449	1	2.208676e-20	1.169914e-23

Table 10: RK and MC solutions of Emden Function of a plane with polytrope index n= 10 and some physical parameters.

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9950412	0.9949823	5.881337e-05	-0.098366399	0.230634525	0.9511339	0.9461699
0.2	0.9806355	0.9805119	1.235893e-04	-0.18754224	0.4398841	0.8218360	0.8053440
0.3	0.9580408	0.9578783	0.0001624934	-0.2613457	0.6131191	0.6507160	0.6228934
0.4	0.9289332	0.9287500	0.0001832605	-0.3176702	0.7453464	0.4778157	0.4434932
0.5	0.8950201	0.8948325	0.0001876149	-0.3578083	0.8395637	0.3293632	0.2945500
0.6	0.8577792	0.8575957	0.0001834766	-0.3848055	0.9028760	0.2152761	0.18454771
0.7	0.8183539	0.8181873	0.0001666102	-0.4020894	0.9433539	0.1344493	0.10999675
0.8	0.7775589	0.7774102	0.0001487076	-0.4126864	0.9681390	0.08059560	0.06268338
0.9	0.7359396	0.7358045	0.0001350636	-0.4189193	0.9827851	0.04647864	0.03422844
1	0.6938432	0.6937208	0.0001223419	-0.4224595	0.9910623	0.02577908	0.017907384
1.1	0.6514810	0.6513661	0.0001149433	-0.4243963	0.9955476	0.013723632	0.008955223

1.2	0.6089767	0.6088682	0.0001085504	-0.4254176	0.9978994	0.006986189	0.004263398
1.3	0.5663997	0.5662976	1.020537e-04	-0.4259284	0.9990584	0.003382418	0.0019208445
1.4	0.5237872	0.5236916	9.561870e-05	-0.4261701	0.9995985	0.0015462709	0.0008125157
1.5	0.4811584	0.4810686	8.986613e-05	-0.4262766	0.9998436	0.0006611931	3.193573e-04
1.6	0.4385226	0.4384389	8.367885e-05	-0.4263210	0.9999454	0.0002612619	1.150801e-04
1.7	0.3958840	0.3958066	7.735581e-05	-0.4263377	0.9999836	9.386691e-05	3.735195e-05
1.8	0.3532443	0.3531732	7.110755e-05	-0.4263440	0.9999955	3.000522e-05	1.066264e-05
1.9	0.3106042	0.3105392	6.501317e-05	-0.4263457	0.9999988	8.279986e-06	2.589890e-06
2.0	0.2679641	0.2679052	5.894024e-05	-0.4263460	0.9999994	1.888331e-06	5.102611e-07
2.2	0.1826839	0.1826371	4.674427e-05	-0.426346	0.9999999	4.074952e-08	7.541873e-09
2.4	0.09740357	0.09736902	3.454271e-05	-0.426346	1	7.461703e-11	7.458112e-12
2.627	0.0001840388	0.0001634059	2.063286e-05	-0.426346	1	1.641728e-40	2.217915e-42

Table 11: The zero and the first derivative of the Emden function calculated by the MC method and the numerical methods of [5] and [1] for the Planar polytrope.

	Emden Radius (t)			The 1st. derv. of the Emden fun. ($y'(t)$)		
n	[5]	[1]	MC	[5]	[1]	MC
0.0	1.414	1.414	1.415	-1.414	-1.414	-1.414
0.5	1.493	1.493	1.493	-1.154	-1.154	-1.155
1	1.570	1.570	1.571	-1.000	-1.000	-0.999
1.5	1.645	1.645	1.645	-0.894	-0.894	-0.894
2	1.717	1.717	1.717	-0.816	-0.816	-0.815
3	1.854	1.854	1.854	-0.707	-0.707	-0.706
4	1.982	1.982	1.982	-0.632	-0.632	-0.632
5	2.103	2.103	2.103	-0.577	-0.577	-0.577
6	2.217	2.217	2.217	-0.534	-0.534	-0.534
10	2.628	2.628	2.627	-0.426	-0.426	-0.426

Table 12: RK and MC solutions of Emden Function of a cylinder with polytrope index n= 0 and some physical parameters

	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
t	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9975001	0.9975084	8.350882e-06	-0.04999977	0.0024999892	1	0.9975084
0.2	0.9900001	0.9900403	4.026914e-05	-0.09999977	0.009999980	1	0.9900403
0.3	0.9775001	0.9775949	9.481533e-05	-0.1499998	0.02249997	1	0.9775949
0.4	0.9600000	0.9601750	0.0001749494	-0.1999998	0.03999996	1	0.9601750
0.5	0.9375000	0.9377491	0.0002490292	-0.2499998	0.06249996	1	0.9377491
0.6	0.9100000	0.9102781	0.0002781076	-0.2999998	0.08999995	1	0.9102781
0.7	0.877500	0.8777773	0.0002773190	-0.3499998	0.12249995	1	0.8777773
0.8	0.8400000	0.8402633	0.0002632527	-0.3999998	0.1599999	1	0.8402633
0.9	0.7975000	0.7977263	0.0002263138	-0.4499998	0.2024999	1	0.7977263
1	0.7500000	0.7502219	0.0002218675	-0.4999998	0.2499999	1	0.7502219
1.1	0.697500	0.6976793	0.0001793378	-0.5499998	0.3024999	1	0.6976793
1.2	0.6400000	0.6401434	0.0001433499	-0.5999998	0.3599999	1	0.6401434
1.3	0.0002219856	0.5776282	0.000128196	-0.6499998	0.4224999	1	0.5775000
1.4	0.5100000	0.5100918	9.182225e-0	-0.6999998	0.4900000	1	0.5100918
1.5	0.4375000	0.4375683	6.830266e-0	-0.7499998	0.5625000	1	0.4375683
1.6	0.3600000	0.3600549	5.493992e-05	-0.7999998	0.6400000	1	0.3600549
1.7	0.2775000	0.2775428	4.281904e-05	-0.8499998	0.7225000	1	0.2775428
1.8	0.1900000	0.1900202	2.020105e-05	-0.8999998	0.8100000	1	0.1900202

1.9	0.09749999	0.09750718	7.197937e-06	-0.9499998	0.9025000	1	0.09750718
2	-1.575266e-08	-1.429902e-05	1.428327e-05	-0.9999998	1.0000000	1	1.429902e-05

Table 13: RK and MC solutions of Emden Function of a cylinder with polytrope index n= 0.5 and some physical parameters

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9975009	0.9974842	1.667389e-05	-0.04993955	3.268525e-03	0.9987413	0.9962287
0.2	0.990012	0.989981	3.111366e-05	-0.09969287	0.013049722	0.9949781	0.9850099
0.3	0.977563	0.9775302	3.321985e-05	-0.1490703	0.02926978	0.9887013	0.9664854
0.4	0.9602005	0.9601660	3.451477e-05	-0.1978849	0.05180597	0.9798806	0.9408481
0.5	0.937990	0.9379371	5.294637e-05	-0.2459346	0.08048164	0.9684715	0.9083654
0.6	0.9110177	0.9109335	8.427058e-05	-0.2930348	0.11507412	0.9544284	0.8694207
0.7	0.8793890	0.8792824	0.0001066372	-0.3389850	0.1553051	0.9377006	0.8245036
0.8	0.8432297	0.8431323	9.740692e-05	-0.3835844	0.2008437	0.9182224	0.7741829
0.9	0.8026866	0.8025907	9.589325e-05	-0.4266127	0.2512948	0.8958743	0.7190204
1	0.757928	0.7578497	7.874448e-05	-0.4678458	0.3062033	0.8705457	0.6597428
1.2	0.6565575	0.6565347	2.280800e-05	-0.5439951	0.4272513	0.8102683	0.5319692
1.4	0.5409510	0.5409720	2.098543e-05	-0.6099116	0.5588587	0.7355080	0.3978892
1.6	0.4134007	0.413437	3.632054e-05	-0.663088	0.6943819	0.6429907	0.2658362
1.8	0.2767755	0.276816	4.108645e-05	-0.7001426	0.8248333	0.5261337	0.1456425
2	0.1347795	0.1347945	1.506449e-05	-0.7157057	0.9368534	0.3671438	0.04948898
2.189	0.0004619707	0.0004590854	2.885296e-06	-0.697986	1.0000000	0.02142628	9.836492e-06

Table 14: RK and MC solutions of Emden Function of a cylinder with polytrope index n= 1 and some physical parameters

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9975018	0.9974860	1.579377e-05	-0.04992371	0.0040001316	0.9974860	0.9949784
0.2	0.9900252	0.9900061	1.913022e-05	-0.09947442	0.015940754	0.9900061	0.9801121
0.3	0.9776265	0.9775850	4.147893e-05	-0.1482845	0.03564385	0.9775850	0.9556725
0.4	0.9603985	0.9603132	8.531098e-05	-0.1959851	0.06281314	0.9603132	0.9222014
0.5	0.9384700	0.9383287	0.0001413654	-0.2422061	0.09703367	0.9383287	0.8804607
0.6	0.9120051	0.9118160	0.0001890526	-0.2866138	0.1377894	0.9118160	0.8314085
0.7	0.8812011	0.8809990	0.0002021483	-0.3288778	0.1844591	0.8809990	0.7761592
0.8	0.8462876	0.8460780	0.0002095765	-0.3687002	0.2363365	0.8460780	0.7158480
0.9	0.8075240	0.8073024	0.0002216093	-0.4058077	0.2926377	0.8073024	0.6517372
1	0.7651979	0.7649612	0.0002367204	-0.4399150	0.3524814	0.7649612	0.5851656
1.2	0.6711329	0.6709064	0.0002265584	-0.4981216	0.4789433	0.6709064	0.4501153
1.4	0.5668553	0.5666349	0.0002203571	-0.5417430	0.6076994	0.5666349	0.3210751
1.6	0.4554023	0.4552032	0.0001990620	-0.5696465	0.7302859	0.4552032	0.2072100
1.8	0.3399865	0.3398362	0.0001502929	-0.5812238	0.8382689	0.3398362	0.11548864
2	0.2238908	0.2237989	9.193208e-05	-0.5764705	0.9237927	0.2237989	0.05008595
2.2	0.11036229	0.11032896	3.333420e-05	-0.5557478	0.9796431	0.11032896	0.012172479
2.204	0.0004286599	0.0004316441	2.984198e-06	-0.5191562	1.0000000	0.0004316441	1.863166e-07

Table 15: RK and MC solutions of Emden Function of a cylinder with polytrope index n= 1.5 and some physical parameters

t	Emden function RK	Emden function MC	Absolute Error (AE)	Solution of the 1st. derv. MC	Physical Parameters: Mass fraction	Density ratio	Pressure ratio
0.1	0.9975023	0.9974884	1.387087e-05	-0.04989826	0.0047019328	0.9962350	0.9937329
0.2	0.9900373	0.9899912	4.609475e-05	-0.09922928	0.018700828	0.9850244	0.9751655
0.3	0.9776885	0.9776228	6.574026e-05	-0.1474553	0.04168431	0.9666227	0.9449924
0.4	0.9605933	0.9605146	7.870501e-05	-0.1940453	0.07313985	0.9413604	0.9041904
0.5	0.9389396	0.9388493	9.031161e-05	-0.2385072	0.11237316	0.9096909	0.8540626
0.6	0.9129629	0.9128716	9.136107e-05	-0.2804003	0.1585334	0.8721969	0.7962037
0.7	0.8829409	0.8828637	7.710879e-05	-0.3193341	0.2106368	0.8295461	0.7323762
0.8	0.8491881	0.8491337	5.432852e-05	-0.3549600	0.2675842	0.7824636	0.6644162
0.9	0.8120501	0.8119954	5.465167e-05	-0.3869995	0.3282041	0.7316955	0.5941333
1	0.7718968	0.7718432	5.365628e-05	-0.4152504	0.3912921	0.6780998	0.5233867
1.2	0.6841065	0.6840675	3.896100e-05	-0.4598468	0.5199785	0.5657811	0.3870325
1.4	0.5890099	0.5889674	4.251962e-05	-0.4883784	0.6442813	0.4519984	0.2662123
1.6	0.4897677	0.4897097	5.802434e-05	-0.5015131	0.7561245	0.3426952	0.1678212
1.8	0.3893226	0.3892674	5.516164e-05	-0.5007459	0.8493388	0.2428690	0.09454098
2	0.2902480	0.2901805	6.748717e-05	-0.4882642	0.9201865	0.1563156	0.04535976
2.2	0.1946367	0.1945692	6.751678e-05	-0.4666376	0.9673717	0.08582444	0.01669879
2.4	0.10402278	0.10398861	3.417447e-05	-0.4388251	0.9924160	0.03353348	0.003487100
2.647	0.0003113408	0.0003074118	3.929012e-06	-0.4009175	1.0000000	5.389900e-06	1.656919e-09

Table 16: RK and MC solutions of Emden Function of a cylinder with polytrope index n= 2 and some physical parameters

t	Emden function RK	Emden function MC	Absolute Error (AE)	Solution of the 1st. derv. MC	Physical Parameters: Mass fraction	Density ratio	Pressure ratio
0.1	0.9975032	0.9974723	3.089961e-05	-0.04986068	5.389678e-03	0.9949509	0.9924359
0.2	0.9900498	0.9900024	4.735432e-05	-0.09897729	0.021397852	0.9801048	0.9703061
0.3	0.9777506	0.9776873	6.331864e-05	-0.1466401	0.04755306	0.9558724	0.9345443
0.4	0.9607860	0.9607043	8.162808e-05	-0.1921707	0.08309057	0.9229528	0.8866848
0.5	0.9394002	0.9393042	9.599690e-05	-0.2349547	0.12698685	0.8822923	0.8287409
0.6	0.9138937	0.9137852	0.0001084830	-0.2744750	0.1780159	0.8350034	0.7630137
0.7	0.8846140	0.8844992	0.0001147942	-0.3103178	0.2348061	0.7823388	0.6919780
0.8	0.8519454	0.8518246	0.0001207948	-0.3421760	0.2958995	0.7256052	0.6180884
0.9	0.8162985	0.8161759	0.0001226248	-0.3698414	0.3598012	0.6661431	0.5436899
1	0.7780991	0.7779676	0.0001315637	-0.3932075	0.4250367	0.6052335	0.4708520
1.2	0.6957619	0.6956333	0.0001285738	-0.4271410	0.5540604	0.4839057	0.3366209
1.4	0.6082754	0.6081636	0.0001118294	-0.4450024	0.6734339	0.3698629	0.2249372
1.6	0.5186642	0.5185561	0.0001080871	-0.4488480	0.7762898	0.2689004	0.1394400
1.8	0.4294669	0.4293620	1.049014e-04	-0.4414012	0.8588367	0.1843517	0.07915361
2	0.3426522	0.3425721	8.016779e-05	-0.4255572	0.9200100	0.1173556	0.04020275
2.2	0.2596070	0.2595357	7.131197e-05	-0.4041682	0.9611462	0.06735877	0.01748200
2.4	0.1811752	0.1811118	6.342567e-05	-0.3797931	0.9852875	0.03280149	0.005940737
2.6	0.10772852	0.10767914	4.938166e-05	-0.3546067	0.9966093	0.011594797	0.0012485178
2.8	0.03924828	0.03923294	1.533466e-05	-0.3303470	0.9998459	0.0015392237	6.038827e-05
2.921	0.0001015976	9.998425e-05	1.613346e-06	-0.3167115	1.0000000	9.996850e-09	9.995276e-13

Table 17: RK and MC solutions of Emden Function of a cylinder with polytrope index n= 3 and some physical parameters

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9975047	0.9974816	2.318186e-05	-0.04977382	6.724411e-03	0.9924637	0.9899642
0.2	0.9900744	0.9900224	5.209039e-05	-0.09843757	0.02659770	0.9703647	0.9606828
0.3	0.9778732	0.9777945	7.869672e-05	-0.1449441	0.05874557	0.9348517	0.9140928
0.4	0.9611638	0.9610731	9.069601e-05	-0.1883728	0.10179619	0.8877062	0.8531505
0.5	0.9402939	0.9402024	9.146140e-05	-0.2279649	0.1539895	0.8311206	0.7814216
0.6	0.9156775	0.9155966	8.087290e-05	-0.2631835	0.2133355	0.7675604	0.7027757
0.7	0.8877755	0.8876960	7.947564e-05	-0.2936502	0.2777036	0.6995082	0.6209507
0.8	0.8570748	0.8569912	8.354217e-05	-0.3192080	0.3449984	0.6294035	0.5393933
0.9	0.8240692	0.8239812	8.797797e-05	-0.3398567	0.4132298	0.5594379	0.4609664
1	0.7892426	0.7891484	9.419084e-05	-0.3557612	0.4806311	0.4914463	0.3878241
1.2	0.7159319	0.7158361	9.579623e-05	-0.3745776	0.6072624	0.3668097	0.2625756
1.4	0.6403682	0.6402807	8.745600e-05	-0.3789243	0.7166940	0.2624891	0.1680667
1.6	0.5650655	0.5649934	7.219359e-05	-0.3725813	0.8053678	0.1803558	0.10189981
1.8	0.4918064	0.4917463	6.008044e-05	-0.3590522	0.8731389	0.1189114	0.05847424
2	0.4217249	0.4216621	6.285741e-05	-0.3412585	0.9220761	0.07497105	0.03161245
2.2	0.3554367	0.3553718	6.493598e-05	-0.3214447	0.9553935	0.04487957	0.01594893
2.4	0.2931752	0.2931053	6.992025e-05	-0.3012109	0.9766417	0.02518088	0.007380647
2.6	0.2349124	0.2348329	7.948975e-05	-0.2815923	0.9891165	0.01295021	0.003041134
2.8	0.1804558	0.1803873	6.850315e-05	-0.2632205	0.9957057	0.005869723	0.0010588232
3	0.1295212	0.1294737	4.752817e-05	-0.2464094	0.9986926	0.002170424	0.0002810129
3.2	0.08178491	0.08175694	2.797383e-05	-0.2312611	0.9997832	0.0005464795	4.467849e-05
3.4	0.03691890	0.03690094	1.796243e-05	-0.2177077	1.0000139	5.024724e-05	1.854170e-06
3.573	0.0001866086	0.0001727404	1.386825e-05	-0.2071637	1.0000000	5.154441e-12	8.903802e-16

Table 18: RK and MC solutions of Emden Function of a cylinder with polytrope index n= 4 and some physical parameters

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9975063	0.9974717	3.460923e-05	-0.04974366	0.0080274839	0.9899250	0.9874222
0.2	0.9900989	0.9900139	8.506663e-05	-0.09800632	0.031631936	0.9606498	0.9510567
0.3	0.9779939	0.9778688	0.0001251394	-0.1434285	0.06943818	0.9143708	0.8941347
0.4	0.9615320	0.9613804	0.0001516465	-0.1848893	0.11934752	0.8542422	0.8212516
0.5	0.9411531	0.9409888	0.0001643044	-0.2215744	0.1787851	0.7840394	0.7377723
0.6	0.9173658	0.9171823	0.0001835758	-0.2529447	0.2449168	0.7076566	0.6490501
0.7	0.8907154	0.8905226	0.0001928146	-0.2787826	0.3149237	0.6288973	0.5600472
0.8	0.8617542	0.8615581	0.0001961507	-0.2991517	0.3862097	0.5509831	0.4747039
0.9	0.8310171	0.8308335	0.0001836283	-0.3143388	0.4565435	0.4764923	0.3958858
1	0.7990016	0.7988405	0.0001611655	-0.3247862	0.5241303	0.4072304	0.3253121
1.2	0.7328709	0.7327492	0.0001217453	-0.3336892	0.6461971	0.2882845	0.2112402
1.4	0.6662536	0.6661703	8.328672e-05	-0.3304862	0.7466604	0.1969433	0.1311978
1.6	0.6011367	0.6010839	5.281985e-05	-0.3195004	0.8249604	0.1305391	0.07846493
1.8	0.5387366	0.5387053	3.129729e-05	-0.3038694	0.8826758	0.08421798	0.04536867
2	0.4796886	0.4796719	1.663745e-05	-0.2861651	0.9236096	0.05293918	0.02539344
2.2	0.4242288	0.4242385	9.728532e-06	-0.2680000	0.9514788	0.03239220	0.01374202
2.4	0.3723394	0.3723876	4.824021e-05	-0.2504132	0.9698626	0.01923007	0.007161042
2.6	0.3238544	0.3239477	9.333680e-05	-0.2338733	0.9812862	0.011012847	0.003567586
2.8	0.2785316	0.2786896	0.0001579731	-0.2187271	0.9883310	0.006032301	0.001681140

3	0.2360989	0.2363326	0.0002336433	-0.2050517	0.9927190	0.003119566	0.0007372550
3.5	0.1408958	0.1411750	0.0002791444	-0.1767427	0.9982776	0.0003972198	5.607750e-05
4.0	0.05829698	0.05842739	0.0001304090	-0.1548202	0.9993768	1.165376e-05	6.808987e-07
4.395	3.742115e-05	3.591943e-05	1.501721e-06	-0.1409936	1.0000000	1.664630e-18	5.979256e-23

Table 19: RK and MC solutions of Emden Function of a cylinder with polytrope index n= 5 and some physical parameters.

	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
t	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9975079	0.9974708	3.703995e-05	-0.04964433	0.0093502759	0.9874179	0.9849205
0.2	0.9901233	0.9900384	8.485863e-05	-0.09746219	0.03671309	0.9511745	0.9416993
0.3	0.9781130	0.9779973	0.0001156315	-0.1418212	0.08013406	0.8947224	0.8750361
0.4	0.9618911	0.9617320	0.0001590541	-0.1814245	0.13668178	0.8227547	0.7912696
0.5	0.9419802	0.9417989	0.0001813301	-0.2154205	0.2028672	0.7409534	0.6978292
0.6	0.9189670	0.9187894	0.0001776016	-0.2433819	0.2750390	0.6547565	0.6015833
0.7	0.8934587	0.8932764	0.0001822997	-0.2653264	0.3498109	0.5687603	0.5080601
0.8	0.8372800	0.8370649	0.0002150980	-0.2925124	0.4241873	0.4866553	0.3439972
0.9	0.8372800	0.8370649	0.0002150980	-0.2925124	0.4958399	0.4109564	0.3439972
1	0.8076453	0.8074051	0.0002402431	-0.2989859	0.5631258	0.3431290	0.2770441
1.2	0.7473722	0.7470866	0.0002855694	-0.3011687	0.6806845	0.2327313	0.1738705
1.4	0.6877373	0.6874135	0.0003237338	-0.2933941	0.7736316	0.1534936	0.10551359
1.6	0.6302784	0.6299384	0.0003399698	-0.2799601	0.8436666	0.09919515	0.06248684
1.8	0.5758077	0.5754920	0.0003156957	-0.2638631	0.8945526	0.06312430	0.03632753
2	0.5246519	0.5243726	0.0002792843	-0.2470163	0.9304871	0.03964604	0.02078930
2.2	0.4768397	0.4766307	0.0002089301	-0.2305393	0.9552620	0.02459859	0.011724442
2.4	0.4322327	0.4320868	0.0001459226	-0.2150276	0.9719866	0.01506104	0.006507676
2.6	0.3906094	0.3905137	9.570355e-05	-0.2007416	0.9830269	0.009081994	0.003546643
2.8	0.3517157	0.3516274	8.832698e-05	-0.1877476	0.9901188	0.005375431	0.001890149
3	0.3152946	0.3151918	1.028772e-04	-0.1760211	0.9945825	0.003110816	0.0009805036
3.5	0.2335062	0.2333713	0.0001348090	-0.1516141	0.9994534	0.0006922096	0.0001615419
4	0.1624657	0.1623064	0.0001592955	-0.1328058	1.000534	1.126364e-04	1.828162e-05
4.5	0.09976912	0.09965379	0.0001153281	-0.1180403	1.000455	9.828090e-06	9.794064e-07
5	0.04368123	0.04363451	4.671525e-05	-0.1062062	1.000172	1.581796e-07	6.902090e-09
5.427	5.635932e-05	0.000037461	1.889832e-05	-0.09783301	1.000000	7.377290e-23	2.763606e-27

Table 20: RK and MC solutions of Emden Function of a cylinder with polytrope index n= 6 and some physical parameters.

	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
t	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9975094	0.9974910	1.836144e-05	-0.04961956	0.0106495818	0.9850404	0.9825690
0.2	0.9901474	0.9900868	6.058942e-05	-0.09704863	0.04165806	0.9419758	0.9326378
0.3	0.9782303	0.9781371	9.318754e-05	-0.1403835	0.09038929	0.8757867	0.8566394
0.4	0.9622414	0.9621084	0.0001329470	-0.1782176	0.1529996	0.7931295	0.7630765
0.5	0.9427771	0.9425826	0.0001945124	-0.2097319	0.2250682	0.7013206	0.6610526
0.6	0.9204883	0.9202614	0.0002269413	-0.2346496	0.3021696	0.6073893	0.5589569
0.7	0.896026	0.8957612	0.0002655311	-0.2532157	0.3804244	0.5165988	0.4627491
0.8	0.8700040	0.8696937	0.0003103668	-0.2659442	0.4566255	0.4327110	0.3763260
0.9	0.8429651	0.8426209	0.0003441502	-0.2736176	0.5285258	0.3579261	0.3015961
1	0.8153738	0.8150104	0.0003634349	-0.2770736	0.5946683	0.2930751	0.2388592
1.2	0.7599766	0.7595941	0.0003824764	-0.2745572	0.7071210	0.1920832	0.1459053
1.4	0.705954	0.7055413	0.0004130184	-0.2640337	0.7933540	0.12334875	0.08702764
1.6	0.6544815	0.6540367	0.0004448203	-0.2495112	0.8568204	0.07827313	0.05119350

1.8	0.606080	0.6056434	0.0004375604	-0.2335545	0.9022781	0.04935166	0.02988951
2	0.5608801	0.5604679	0.0004121400	-0.2176373	0.9342068	0.03099592	0.01737222
2.5	0.4610245	0.4605964	0.0004281327	-0.1820780	0.9769607	0.009548237	0.004397884
3	0.3770324	0.3766617	0.0003706990	-0.1542081	0.9929057	0.002855674	0.0010756229
3.5	0.3052820	0.3050481	0.0002339380	-0.1329092	0.9983957	0.0008057679	0.0002457980
4	0.242916	0.2428582	5.834478e-05	-0.1164880	1.0000478	0.0002051714	4.982757e-05
4.5	0.1878506	0.1878574	6.758868e-06	-0.1035982	1.000563	4.395110e-05	8.256539e-06
5	0.138580	0.1385512	2.877910e-05	-0.09325589	1.000751	7.073944e-06	9.801033e-07
5.5	0.09400703	0.09394591	6.112392e-05	-0.08476922	1.000646	6.874914e-07	6.458700e-08
6	0.05331487	0.05324670	6.817172e-05	-0.07768225	1.000352	2.279062e-08	1.213525e-09
6.5	0.01588170	0.01586806	1.363838e-05	-0.07168862	1.000100	1.596404e-11	2.533182e-13
6.724	3.671932e-05	0.000054544	1.782468e-05	-0.06929352	1.000000	2.633188e-26	1.436246e-30

Table 21: RK and MC solutions of Emden Function of a cylinder with polytrope index n= 10 and some physical parameters

t	Emden function		Absolute Error (AE)	Solution of the 1st. derv.	Physical Parameters		
	RK	MC		MC	Mass fraction	Density ratio	Pressure ratio
0.1	0.9973439	0.9959388	0.001405112	-0.05469767	0.017526375	0.9601225	0.9562233
0.2	0.9900567	0.9882679	0.001788839	-0.09662910	0.06192431	0.8886827	0.8782565
0.3	0.9785028	0.9765447	0.001958100	-0.1342410	0.12904154	0.7887171	0.7702175
0.4	0.9633974	0.9614455	0.001951917	-0.1645758	0.2109352	0.6749111	0.6488902
0.5	0.9455559	0.9437177	0.001838164	-0.1870618	0.2996941	0.5603008	0.5287658
0.6	0.9257808	0.9241346	0.001646237	-0.2021976	0.3887322	0.4543098	0.4198434
0.7	0.9047859	0.9033620	0.001423940	-0.2110212	0.4733118	0.3619246	0.3269490
0.8	0.8831599	0.8819727	0.001187288	-0.2147917	0.5505930	0.2848074	0.2511923
0.9	0.8613595	0.8604093	0.0009502683	-0.2147232	0.6192198	0.2223570	0.1913180
1	0.8397209	0.8390014	0.0007195756	-0.2118987	0.6789715	0.1728330	0.1450071
1.2	0.7977921	0.7974900	0.0003021302	-0.2012539	0.7738356	0.10405243	0.08298076
1.4	0.7584188	0.7584778	5.900775e-05	-0.1875971	0.8415449	0.06301279	0.04779380
1.6	0.7219201	0.7222523	0.0003322430	-0.1734344	0.8891569	0.03862685	0.02789833
1.8	0.6882535	0.6887875	0.0005339870	-0.1599400	0.9224714	0.02403546	0.01655533
2	0.6572191	0.6579282	0.0007090995	-0.1475867	0.9458022	0.01519795	0.009999161
3	0.5328848	0.5336934	0.0008086413	-0.1032272	0.9922900	0.001874642	0.0010004839
4	0.4425483	0.4431262	0.0005778692	-0.07813363	1.001432	0.0002919271	0.0001293605
5	0.3721522	0.3726724	0.0005201459	-0.06265702	1.003836	5.167398e-05	1.925746e-05
6	0.3145769	0.3151344	0.0005575371	-0.05226564	1.004826	9.659589e-06	3.044069e-06
7	0.2658869	0.2664116	0.0005246858	-0.04483219	1.005568	1.801070e-06	4.798259e-07
8	0.2237076	0.2241748	0.0004672052	-0.03923226	1.005672	3.205297e-07	7.185468e-08
9	0.1865025	0.1869601	0.0004576417	-0.03487218	1.005645	5.217833e-08	9.755268e-09
10	0.1532213	0.1536215	0.0004002091	-0.03136856	1.005120	7.320167e-09	1.124535e-09
11	0.1231148	0.1234383	0.0003234798	-0.02849606	1.004386	8.212962e-10	1.013794e-10
12	0.09562976	0.09577904	0.0001492776	-0.02609248	1.003275	6.496878e-11	6.222647e-12
13	0.07034594	0.07036817	2.222264e-05	-0.02405969	1.002205	2.976887e-12	2.094781e-13
14	0.04693679	0.04695146	1.467487e-05	-0.02232047	1.001278	5.205842e-14	2.444219e-15
15	0.02514338	0.02513700	6.378203e-06	-0.02082082	1.000720	1.007244e-16	2.531910e-18
16	0.004757011	0.004694068	6.294338e-05	-0.01950781	1.000119	5.193902e-24	2.438053e-26
16.222	0.0004043166	0.000332760	7.155658e-05	-0.01923855	1.000000	1.664605e-35	5.539139e-39

Table 22: The zero and the first derivative of the Emden function calculated by the MC method and the numerical methods of [5] and [1] for the cylindrical polytropes.

	Emden Radius (t)			The 1st. derv. of the Emden fun. ($y'(t)$)		
n	[5]	[1]	MC	[5]	[1]	MC
0.0	2.000	2.000	2	-1.000	-1.000	-0.999
0.5	2.189	2.189	2.189	-0.697	-0.697	-0.697
1	2.404	2.404	2.204	-0.519	-0.519	-0.519
1.5	2.647	2.647	2.647	-0.400	-0.400	-0.400
2	2.921	2.921	2.921	-0.316	-0.316	-0.316
3	3.573	3.573	3.573	-0.207	-0.207	-0.207
4	4.395	4.395	4.395	-0.140	-0.140	-0.140
5	5.427	5.427	5.427	-0.098	-0.098	-0.097
6	6.724	6.724	6.724	-0.695	-0.695	-0.069
10	16.222	16.222	16.222	-0.019	-0.019	-0.019