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Tripled Fixed Point Theorems with Generalized Contractions in Hilbert Spaces

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Abstract:

The purpose of this manuscript is to find the existence and uniqueness of triple fixed point results in the setting of Hilbert spaces with implicit relations. Moreover, the well-posedness of the tripled fixed point problem of self-mappings in Hilbert spaces is discussed. Consequently, to promote our paper, an illustrative example is derived.
keywords: Hilbert spaces, tripled fixed points, asymptotically regular sequences, implicit relations, well-posedness.

1 Introduction

After the emergence of Banach's theorem [1], the technique and the applications of fixed point (FP) became very important in several fields of mathematics, statistics, computer science, engineering, economics, biology, game theory, chemistry, theory of differential equations, theory of integral equations, theory of matrix equations, mathematical economics, etc. (see, [2,3,4,5,6]).

In 2011, the notion of triple fixed points (TFPs) was initiated by Berinde and Borcut [7]. Moreover, many authors presented some TFP results for contractive mappings in several spaces, for more contributions in this regard, (see [8,9,10,11,12]).

Inner product spaces are a sub-kind of normed spaces which are older than ordinary normed spaces. Their theory is more comprehensive and retains many aspects of Euclidean space. D. Hilbert presented good work in these special spaces. Many authors have started working in this field using FP theory techniques (see [13,14,15, 16,17]). Blassi and Myjak [18] defined another approach to FPs in 1989, called well-posedness of FP problem. Recently, some results in Hilbert spaces (HSs) using coupled implicit relations are presented by S. Kim [19], who discussed the well-posedness of a coupled FP problem. The concept of well-posedness of a FP problem for a single-valued mapping was established by several authors, see [20,21,22,23]. Following the previous literatures, we introduce TPF theorems for self-mapping in HS and present feasible conditions for the existence of solutions to TFP problems, and discuss a technique for guaranteeing the well-posedness of TFP problems

2 Preliminaries

In this section, we present some notations and basic definitions which are useful in this manuscript.

Definition 1.*Assume that Ω is a real linear vector space with field $\mathbb{R}$.. Suppose $\|.\| : \Omega \to \mathbb{R}^+$ satisfies*

(1)$\|\beta\| \geq 0$;
(2)$\|\beta\| = 0$ *if and only if* $\beta = 0$,
(3)$\|\sigma\beta\| = |\sigma| \, \|\beta\|$,
(4)$\|\beta+\zeta\| \leq \|\beta\| + \|\zeta\|$,

for all $\beta, \zeta \in \Omega$ and scalers $\sigma \in \mathbb{R}$. Then $\|.\|$ is called a norm and $(\Omega, \|.\|)$ is called a linear normed space.

Remark. An inner product on Ω describes a norm on it as

$$\|\beta\| = \sqrt{\langle \beta, \beta \rangle}.$$

Definition 2.*Suppose that $(\Omega, \|.\|)$ is called a linear normed space. Then,*

(i)*A sequence $\{\beta_\omega\}$ converges to a point $\beta \in \Omega$ if*

$$\lim_{\omega\to\infty} \|\beta_\omega - \beta\| = 0,$$

for all $\omega \in \mathbb{N}$.

(ii)*A sequence* $\{\beta_\omega\}$ *is called a Cauchy sequence* if

$$\lim_{\omega,\rho\to\infty} \|\beta_\omega - \beta_\rho\| = 0,$$

for all $\omega, \rho \in \mathbb{N}$. We say that $(\Omega, \|.\|)$ is a Banach space if every Cauchy sequence is convergent in Ω

*Remark.*HSs are Banach spaces.

Definition 3.*We say that a mapping* $\Gamma : \Omega^3 \to \Omega$ *is continuous at some* $(\beta, \zeta, \delta) \in \Omega^3$ *if for any sequences*

$\{\beta_\omega\}, \{\zeta_\omega\}$ *and* $\{\delta_\omega\}$, *we get*

$$\Gamma(\beta_\omega, \zeta_\omega, \delta_\omega) \to \Gamma(\beta, \zeta, \delta),$$

$$\Gamma(\zeta_\omega, \delta_\omega, \beta_\omega) \to \Gamma(\zeta, \delta, \beta),$$

and

$$\Gamma(\delta_\omega, \beta_\omega, \zeta_\omega) \to \Gamma(\delta, \beta, \zeta), \text{ as } \omega \to \infty,$$

where $\beta_\omega \to \beta$, $\zeta_\omega \to \zeta$ and $\delta_\omega \to \delta$.

In 2011, Berinde and Borcut [7] presented the idea of TFPs as a generalization of coupled FPs. Some basic notions concerning TFPs are given below:

Definition 4.*[7] Let* $\Gamma : \Omega^3 \to \Omega$ *be a mapping. A point* $(\beta, \zeta, \delta) \in \Omega^3$ *is called a TFP of* Γ *if*

$$\beta = \Gamma(\beta, \zeta, \delta),$$

$$\zeta = \Gamma(\zeta, \delta, \beta),$$

and

$$\delta = \Gamma(\delta, \beta, \zeta).$$

The set of all TFPs of Γ *in* Ω^3 *is denoted by* $T\left(\Gamma, \Omega^3\right)$.

Definition 5.*[12] Let* $\Gamma, \Xi : \Omega^3 \to \Omega$ *be mappings. A point* $(\beta, \zeta, \delta) \in \Omega^3$ *is called a common TFP of* Γ *and* Ξ *if*

$$\Gamma(\beta, \zeta, \delta) = \beta = \Xi(\beta, \zeta, \delta),$$

$$\Gamma(\zeta, \delta, \beta) = \zeta = \Xi(\zeta, \delta, \beta),$$

and

$$\Gamma(\delta, \beta, \zeta) = \delta = \Xi(\delta, \beta, \zeta).$$

Example 1.Let $\Omega = [0, \infty)$ *and* $\Gamma : \Omega^3 \to \Omega$ *be a mapping given by*

$$\Gamma(\beta, \zeta, \delta) = \frac{\beta + \zeta + \delta}{3}, \text{ for all } \beta, \zeta, \delta \in \Omega.$$

Then there is a unique TFP of Ω, *whenever* $\beta = \zeta = \delta$.

Definition 6.*A sequence* $\{\beta_\omega\}$ *is said to be an asymptotically* $\Gamma-$*regular sequence in HS if it satisfies*

$$\lim_{\omega\to\infty} \|\beta_\omega - \Gamma(\beta_\omega)\| = 0.$$

Definition 7.*Assume that* $\{\beta_\omega\}, \{\zeta_\omega\}$ *and* $\{\delta_\omega\}$ *are three sequences in HS, then the triple* $(\{\beta_\omega\}, \{\zeta_\omega\}, \{\delta_\omega\}) \in \Omega^3$ *is said to be a tripled asymptotically* $\Gamma-$*regular if,*

$$\lim_{\omega\to\infty} \|\beta_\omega - \Gamma(\beta_\omega, \zeta_\omega, \delta_\omega)\| = 0,$$

$$\lim_{\omega\to\infty} \|\zeta_\omega - \Gamma(\zeta_\omega, \delta_\omega, \beta_\omega)\| = 0,$$

and

$$\lim_{\omega\to\infty} \|\delta_\omega - \Gamma(\delta_\omega, \beta_\omega, \zeta_\omega)\| = 0.$$

The notion of implicit relation in the framework of FP theory was introduced by V. Popa [24] in 1999. Now we introduce a new contribution for such relation as follows:

Definition 8.*Suppose that* $\varphi : \mathbb{R}^6 \to \mathbb{R}^+$ *is a continuous function and suppose it is non-decreasing in the fifth and the sixth arguments, we say the following relation is a tripled implicit relation for any* $\beta, \zeta, \delta, \mu_1, \mu_2, \mu_3 > 0$ *if,*

(i)

$$\beta \le \varphi\left(\frac{\mu_1+\mu_2+\mu_3}{3}, \frac{\beta+\mu_1}{3}, \frac{\zeta+\mu_2}{3}, \frac{\delta+\mu_3}{3}, \zeta+\mu_2, \delta+\mu_3\right),$$

$$\zeta \le \varphi\left(\frac{\mu_1+\mu_2+\mu_3}{3}, \frac{\zeta+\mu_2}{3}, \frac{\delta+\mu_3}{3}, \frac{\beta+\mu_1}{3}, \delta+\mu_3, \beta+\mu_1\right),$$

and

$$\delta \le \varphi\left(\frac{\mu_1+\mu_2+\mu_3}{3}, \frac{\delta+\mu_3}{3}, \frac{\beta+\mu_1}{3}, \frac{\zeta+\mu_2}{3}, \beta+\mu_1, \zeta+\mu_2\right).$$

or

(ii)

$$\beta \le \varphi\left(\frac{\mu_1+\mu_2+\mu_3}{3}, 0, 0, 0, \mu_2, \mu_3\right),$$

$$\zeta \le \varphi\left(\frac{\mu_1+\mu_2+\mu_3}{3}, 0, 0, 0, \mu_3, \mu_1\right),$$

and

$$\delta \le \varphi\left(\frac{\mu_1+\mu_2+\mu_3}{3}, 0, 0, 0, \mu_1, \mu_2\right).$$

Then, there exists a real number $\alpha \in (0, 1)$ such that $\beta + \zeta + \delta \le \alpha(\mu_1 + \mu_2 + \mu_3)$.

From now, every function fulfills this implicit relation will be a member of $\Phi-$*family.*

Definition 9. *In HS we say that the pair (Γ,Ξ) satisfies a $\varphi-$contraction if for all $\beta,\zeta,\delta,\beta^*,\zeta^*,\delta^*\in\Omega$, one can write*

$$\begin{aligned}
&\|\Gamma(\beta,\zeta,\delta)-\Xi(\beta^*,\zeta^*,\delta^*)\|^2\\
&\leq\varphi\Bigg(\frac{\|\beta-\beta^*\|^2+\|\zeta-\zeta^*\|^2+\|\delta-\delta^*\|^2}{3},\\
&\frac{\|\beta-\Gamma(\beta,\zeta,\delta)\|^2+\|\beta^*-\Xi(\beta^*,\zeta^*,\delta^*)\|^2}{3},\\
&\frac{\|\zeta-\Gamma(\zeta,\delta,\beta)\|^2+\|\zeta^*-\Xi(\zeta^*,\delta^*,\beta^*)\|^2}{3},\\
&\frac{\|\delta-\Gamma(\delta,\beta,\zeta)\|^2+\|\delta^*-\Xi(\delta^*,\beta^*,\zeta^*)\|^2}{3},\\
&\frac{\|\zeta^*-\Xi(\zeta,\delta,\beta)\|^2+\|\zeta-\Gamma(\zeta^*,\delta^*,\beta^*)\|^2}{3},\\
&\frac{\|\delta^*-\Xi(\delta,\beta,\zeta)\|^2+\|\delta-\Gamma(\delta^*,\beta^*,\zeta^*)\|^2}{3}\Bigg), \qquad (1)
\end{aligned}$$

where $\varphi\in\Phi-$family.

Lemma 1. *[19] Assume that Ω is HS, then for any positive integer ς, we obtain*

$$(\beta_1+\beta_2+\cdots+\beta_\varsigma)^2\leq\varsigma\left(\beta_1^2+\beta_2^2+\cdots+\beta_\varsigma^2\right),$$

for all $\beta_\omega\in\Omega$ where $\omega=1,2,3,\cdots,\varsigma$.

3 Main results

In this section, we present our main results for the existence and uniqueness of TFPs of a self-mapping by using rational type contractions equipped with implicit relation. Also, we introduce a fine condition for locating TFPs for a sequence of self-mappings in HSs.

Theorem 1. *Let $\eth$ be a closed subset of a HS Ω, and $\Gamma,\Xi:\eth^3\to\eth$, such that*

(1) Γ and Ξ are continuous,

(2) the pair (Γ,Ξ) justifies $\varphi-$contraction.

Then, Γ and Ξ have a common TFP in $\eth^3$.

Proof. Define $\beta_0,\zeta_0,\delta_0\in\eth$ such that for $\Gamma,\Xi:\eth^3\to\eth$, we have

$$\beta_1=\Gamma(\beta_0,\zeta_0,\delta_0),\ \zeta_1=\Gamma(\zeta_0,\delta_0,\beta_0),\ \delta_1=\Gamma(\delta_0,\beta_0,\zeta_0),$$

and

$$\beta_2=\Xi(\beta_1,\zeta_1,\delta_1),\ \zeta_2=\Xi(\zeta_1,\delta_1,\beta_1),\ \delta_2=\Xi(\delta_1,\beta_1,\zeta_1),$$

iteratively, we obtain

$$\begin{aligned}
\beta_{\omega+1}&=\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega),\\
\zeta_{\omega+1}&=\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega),\\
\delta_{\omega+1}&=\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega),
\end{aligned}$$

and

$$\begin{aligned}
\beta_{\omega+2}&=\Xi(\beta_{\omega+1},\zeta_{\omega+1},\delta_{\omega+1}),\\
\zeta_{\omega+2}&=\Xi(\zeta_{\omega+1},\delta_{\omega+1},\beta_{\omega+1}),\\
\delta_{\omega+2}&=\Xi(\delta_{\omega+1},\beta_{\omega+1},\zeta_{\omega+1}).
\end{aligned}$$

Consider

$$\begin{aligned}
&\|\beta_{\omega+1}-\beta_\omega\|^2\\
&=\|\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\Xi(\beta_{\omega-1},\zeta_{\omega-1},\delta_{\omega-1})\|^2\\
&\leq\varphi\Bigg(\frac{\|\beta_\omega-\beta_{\omega-1}\|^2+\|\zeta_\omega-\zeta_{\omega-1}\|^2+\|\delta_\omega-\delta_{\omega-1}\|^2}{3},\\
&\frac{1}{3}\begin{pmatrix}\|\beta_\omega-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2\\+\|\beta_{\omega-1}-\Xi(\beta_{\omega-1},\zeta_{\omega-1},\delta_{\omega-1})\|^2\end{pmatrix},\\
&\frac{1}{3}\begin{pmatrix}\|\zeta_\omega-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2\\+\|\zeta_{\omega-1}-\Xi(\zeta_{\omega-1},\delta_{\omega-1},\beta_{\omega-1})\|^2\end{pmatrix},\\
&\frac{1}{3}\begin{pmatrix}\|\delta_\omega-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2\\+\|\delta_{\omega-1}-\Xi(\delta_{\omega-1},\beta_{\omega-1},\zeta_{\omega-1})\|^2\end{pmatrix},\\
&\frac{1}{3}\begin{pmatrix}\|\zeta_{\omega-1}-\Xi(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2\\+\|\zeta_\omega-\Gamma(\zeta_{\omega-1},\delta_{\omega-1},\beta_{\omega-1})\|^2\end{pmatrix},\\
&\frac{1}{3}\begin{pmatrix}\|\delta_{\omega-1}-\Xi(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2\\+\|\delta_\omega-\Gamma(\delta_{\omega-1},\beta_{\omega-1},\zeta_{\omega-1})\|^2\end{pmatrix}\Bigg)
\end{aligned}$$

$$\begin{aligned}
&=\varphi\Bigg(\frac{\|\beta_\omega-\beta_{\omega-1}\|^2+\|\zeta_\omega-\zeta_{\omega-1}\|^2+\|\delta_\omega-\delta_{\omega-1}\|^2}{3},\\
&\frac{\|\beta_\omega-\beta_{\omega+1}\|^2+\|\beta_{\omega-1}-\beta_\omega\|^2}{3},\\
&\frac{\|\zeta_\omega-\zeta_{\omega+1}\|^2+\|\zeta_{\omega-1}-\zeta_\omega\|^2}{3},\\
&\frac{\|\delta_\omega-\delta_{\omega+1}\|^2+\|\delta_{\omega-1}-\delta_\omega\|^2}{3},\\
&\frac{\|\zeta_{\omega-1}-\zeta_{\omega+1}\|^2+\|\zeta_\omega-\zeta_\omega\|^2}{3},\\
&\frac{\|\delta_{\omega-1}-\delta_{\omega+1}\|^2+\|\delta_\omega-\delta_\omega\|^2}{3}\Bigg)
\end{aligned}$$

$$\leq \left(\frac{\|\beta_\omega-\beta_{\omega-1}\|^2+\|\zeta_\omega-\zeta_{\omega-1}\|^2+\|\delta_\omega-\delta_{\omega-1}\|^2}{3}, \frac{\|\beta_\omega-\beta_{\omega+1}\|^2+\|\beta_{\omega-1}-\beta_\omega\|^2}{3}, \frac{\|\zeta_\omega-\zeta_{\omega+1}\|^2+\|\zeta_{\omega-1}-\zeta_\omega\|^2}{3}, \frac{\|\delta_\omega-\delta_{\omega+1}\|^2+\|\delta_{\omega-1}-\delta_\omega\|^2}{3}, \|\zeta_\omega-\zeta_{\omega-1}\|^2+\|\zeta_\omega-\zeta_{\omega+1}\|^2, \|\delta_\omega-\delta_{\omega-1}\|^2+\|\delta_\omega-\delta_{\omega+1}\|^2, \right)$$

Similarly, we can obtain

$$\begin{aligned} &\|\zeta_{\omega+1}-\zeta_\omega\|^2 \\ &= \|\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)-\Xi(\zeta_{\omega-1},\delta_{\omega-1},\beta_{\omega-1})\|^2 \\ &\leq \varphi\left(\frac{\|\zeta_\omega-\zeta_{\omega-1}\|^2+\|\delta_\omega-\delta_{\omega-1}\|^2+\|\beta_\omega-\beta_{\omega-1}\|^2}{3}, \right. \\ &\frac{\|\zeta_\omega-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2+\|\zeta_{\omega-1}-\Xi(\zeta_{\omega-1},\delta_{\omega-1},\beta_{\omega-1})\|^2}{3}, \\ &\frac{\|\delta_\omega-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2+\|\delta_{\omega-1}-\Xi(\delta_{\omega-1},\beta_{\omega-1},\zeta_{\omega-1})\|^2}{3}, \\ &\frac{\|\beta_\omega-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2+\|\beta_{\omega-1}-\Xi(\beta_{\omega-1},\zeta_{\omega-1},\delta_{\omega-1})\|^2}{3}, \\ &\frac{\|\delta_{\omega-1}-\Xi(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2+\|\delta_\omega-\Gamma(\delta_{\omega-1},\beta_{\omega-1},\zeta_{\omega-1})\|^2}{3}, \\ &\left. \frac{\|\beta_{\omega-1}-\Xi(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2+\|\beta_\omega-\Gamma(\beta_{\omega-1},\zeta_{\omega-1},\delta_{\omega-1})\|^2}{3} \right) \end{aligned}$$

$$\begin{aligned} &= \varphi\left(\frac{\|\zeta_\omega-\zeta_{\omega-1}\|^2+\|\delta_\omega-\delta_{\omega-1}\|^2+\|\beta_\omega-\beta_{\omega-1}\|^2}{3}, \right. \\ &\frac{\|\zeta_\omega-\zeta_{\omega+1}\|^2+\|\zeta_{\omega-1}-\zeta_\omega\|^2}{3}, \\ &\frac{\|\delta_\omega-\delta_{\omega+1}\|^2+\|\delta_{\omega-1}-\delta_\omega\|^2}{3}, \\ &\frac{\|\beta_\omega-\beta_{\omega+1}\|^2+\|\beta_{\omega-1}-\beta_\omega\|^2}{3}, \\ &\frac{\|\delta_{\omega-1}-\delta_{\omega+1}\|^2+\|\delta_\omega-\delta_\omega\|^2}{3}, \\ &\left. \frac{\|\beta_{\omega-1}-\beta_{\omega+1}\|^2+\|\beta_\omega-\beta_\omega\|^2}{3} \right) \end{aligned}$$

$$\leq \left(\frac{\|\zeta_\omega-\zeta_{\omega-1}\|^2+\|\delta_\omega-\delta_{\omega-1}\|^2+\|\beta_\omega-\beta_{\omega-1}\|^2}{3}, \frac{\|\zeta_\omega-\zeta_{\omega+1}\|^2+\|\zeta_{\omega-1}-\zeta_\omega\|^2}{3}, \frac{\|\delta_\omega-\delta_{\omega+1}\|^2+\|\delta_{\omega-1}-\delta_\omega\|^2}{3}, \frac{\|\beta_\omega-\beta_{\omega+1}\|^2+\|\beta_{\omega-1}-\beta_\omega\|^2}{3}, \|\delta_\omega-\delta_{\omega-1}\|^2+\|\delta_\omega-\delta_{\omega+1}\|^2, \|\beta_\omega-\beta_{\omega-1}\|^2+\|\beta_\omega-\beta_{\omega+1}\|^2, \right)$$

Again, similarly

$$\begin{aligned} &\|\delta_{\omega+1}-\delta_\omega\|^2 \\ &= \|\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)-\Xi(\delta_{\omega-1},\beta_{\omega-1},\zeta_{\omega-1})\|^2 \\ &\leq \varphi\left(\frac{\|\delta_\omega-\delta_{\omega-1}\|^2+\|\beta_\omega-\beta_{\omega-1}\|^2+\|\zeta_\omega-\zeta_{\omega-1}\|^2}{3}, \right. \\ &\frac{1}{3}\left(\begin{array}{c} \|\delta_\omega-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2 \\ +\|\delta_{\omega-1}-\Xi(\delta_{\omega-1},\beta_{\omega-1},\zeta_{\omega-1})\|^2 \end{array} \right), \\ &\frac{1}{3}\left(\begin{array}{c} \|\beta_\omega-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2 \\ +\|\beta_{\omega-1}-\Xi(\beta_{\omega-1},\zeta_{\omega-1},\delta_{\omega-1})\|^2 \end{array} \right), \\ &\frac{1}{3}\left(\begin{array}{c} \|\zeta_\omega-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2 \\ +\|\zeta_{\omega-1}-\Xi(\zeta_{\omega-1},\delta_{\omega-1},\beta_{\omega-1})\|^2 \end{array} \right), \\ &\frac{1}{3}\left(\begin{array}{c} \|\beta_{\omega-1}-\Xi(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2 \\ +\|\beta_\omega-\Gamma(\beta_{\omega-1},\zeta_{\omega-1},\delta_{\omega-1})\|^2 \end{array} \right), \\ &\left. \frac{1}{3}\left(\begin{array}{c} \|\zeta_{\omega-1}-\Xi(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2 \\ +\|\zeta_\omega-\Gamma(\zeta_{\omega-1},\delta_{\omega-1},\beta_{\omega-1})\|^2 \end{array} \right) \right) \end{aligned}$$

$$\begin{aligned} &= \varphi\left(\frac{\|\delta_\omega-\delta_{\omega-1}\|^2+\|\beta_\omega-\beta_{\omega-1}\|^2+\|\zeta_\omega-\zeta_{\omega-1}\|^2}{3}, \right. \\ &\frac{\|\delta_\omega-\delta_{\omega+1}\|^2+\|\delta_{\omega-1}-\delta_\omega\|^2}{3}, \\ &\frac{\|\beta_\omega-\beta_{\omega+1}\|^2+\|\beta_{\omega-1}-\beta_\omega\|^2}{3}, \\ &\frac{\|\zeta_\omega-\zeta_{\omega+1}\|^2+\|\zeta_{\omega-1}-\zeta_\omega\|^2}{3}, \\ &\frac{\|\beta_{\omega-1}-\beta_{\omega+1}\|^2+\|\beta_\omega-\beta_\omega\|^2}{3}, \\ &\left. \frac{\|\zeta_{\omega-1}-\zeta_{\omega+1}\|^2+\|\zeta_\omega-\zeta_\omega\|^2}{3} \right) \end{aligned}$$

$$\leq \left(\frac{\|\delta_\omega - \delta_{\omega-1}\|^2 + \|\beta_\omega - \beta_{\omega-1}\|^2 + \|\zeta_\omega - \zeta_{\omega-1}\|^2}{3}, \frac{\|\delta_\omega - \delta_{\omega+1}\|^2 + \|\delta_{\omega-1} - \delta_\omega\|^2}{3}, \frac{\|\beta_\omega - \beta_{\omega+1}\|^2 + \|\beta_{\omega-1} - \beta_\omega\|^2}{3}, \frac{\|\zeta_\omega - \zeta_{\omega+1}\|^2 + \|\zeta_{\omega-1} - \zeta_\omega\|^2}{3}, \|\beta_\omega - \beta_{\omega-1}\|^2 + \|\beta_\omega - \beta_{\omega+1}\|^2, \|\zeta_\omega - \zeta_{\omega-1}\|^2 + \|\zeta_\omega - \zeta_{\omega+1}\|^2 \right).$$

Using hypothesis (i) of Definition 8, one can write the first property of $\varphi \in \Phi-$family, there is $\alpha \in (0,1)$ such that

$$\|\beta_{\omega+1} - \beta_\omega\|^2 + \|\zeta_{\omega+1} - \zeta_\omega\|^2 + \|\delta_{\omega+1} - \delta_\omega\|^2 \leq \alpha \left(\|\beta_\omega - \beta_{\omega-1}\|^2 + \|\zeta_\omega - \zeta_{\omega-1}\|^2 + \|\delta_\omega - \delta_{\omega-1}\|^2 \right).$$

With the same manner, we get

$$\|\beta_{\omega+1} - \beta_\omega\|^2 + \|\zeta_{\omega+1} - \zeta_\omega\|^2 + \|\delta_{\omega+1} - \delta_\omega\|^2 \leq \alpha^2 \left(\|\beta_{\omega-1} - \beta_{\omega-2}\|^2 + \|\zeta_{\omega-1} - \zeta_{\omega-2}\|^2 + \|\delta_{\omega-1} - \delta_{\omega-2}\|^2 \right).$$

Iteratively,

$$\|\beta_{\omega+1} - \beta_\omega\|^2 + \|\zeta_{\omega+1} - \zeta_\omega\|^2 + \|\delta_{\omega+1} - \delta_\omega\|^2 \leq \alpha^\omega \left(\|\beta_1 - \beta_0\|^2 + \|\zeta_1 - \zeta_0\|^2 + \|\delta_1 - \delta_0\|^2 \right).$$

Using Lemma 1 and the triangular inequality for any positive integer ς, we have

$$\begin{aligned} &\left(\|\beta_\omega - \beta_{\omega+\varsigma}\|^2 + \|\zeta_\omega - \zeta_{\omega+\varsigma}\|^2 + \|\delta_\omega - \delta_{\omega+\varsigma}\|^2 \right) \\ \leq& \left(\|\beta_\omega - \beta_{\omega+1}\| + \|\beta_{\omega+1} - \beta_{\omega+2}\| + \cdots + \|\beta_{\omega+\varsigma-1} - \beta_{\omega+\varsigma}\| \right)^2 \\ &+ \left(\|\zeta_\omega - \zeta_{\omega+1}\| + \|\zeta_{\omega+1} - \zeta_{\omega+2}\| + \cdots + \|\zeta_{\omega+\varsigma-1} - \zeta_{\omega+\varsigma}\| \right)^2 \\ &+ \left(\|\delta_\omega - \delta_{\omega+1}\| + \|\delta_{\omega+1} - \delta_{\omega+2}\| + \cdots + \|\delta_{\omega+\varsigma-1} - \delta_{\omega+\varsigma}\| \right)^2 \\ \leq& \varsigma \Bigg[\left(\|\beta_\omega - \beta_{\omega+1}\|^2 + \|\beta_{\omega+1} - \beta_{\omega+2}\|^2 + \cdots + \|\beta_{\omega+\varsigma-1} - \beta_{\omega+\varsigma}\|^2 \right) \\ &+ \left(\|\zeta_\omega - \zeta_{\omega+1}\|^2 + \|\zeta_{\omega+1} - \zeta_{\omega+2}\|^2 + \cdots + \|\zeta_{\omega+\varsigma-1} - \zeta_{\omega+\varsigma}\|^2 \right) \\ &+ \left(\|\delta_\omega - \delta_{\omega+1}\|^2 + \|\delta_{\omega+1} - \delta_{\omega+2}\|^2 + \cdots + \|\delta_{\omega+\varsigma-1} - \delta_{\omega+\varsigma}\|^2 \right) \Bigg]. \end{aligned}$$

$$\begin{aligned} =& \varsigma \Bigg[\left(\|\beta_\omega - \beta_{\omega+1}\|^2 + \|\zeta_\omega - \zeta_{\omega+1}\|^2 + \|\delta_\omega - \delta_{\omega+1}\|^2 \right) \\ &+ \left(\|\beta_{\omega+1} - \beta_{\omega+2}\|^2 + \|\zeta_{\omega+1} - \zeta_{\omega+2}\|^2 + \|\delta_{\omega+1} - \delta_{\omega+2}\|^2 \right) \\ &+ \cdots \\ &+ \left(\|\beta_{\omega+\varsigma-1} - \beta_{\omega+\varsigma}\|^2 + \|\zeta_{\omega+\varsigma-1} - \zeta_{\omega+\varsigma}\|^2 + \|\delta_{\omega+\varsigma-1} - \delta_{\omega+\varsigma}\|^2 \right) \Bigg]. \end{aligned}$$

$$\begin{aligned} \leq& \varsigma \left(\alpha^\omega + \alpha^{\omega+1} + \cdots + \alpha^{\omega+\varsigma-1} \right) \\ &\times \left(\|\beta_1 - \beta_0\|^2 + \|\zeta_1 - \zeta_0\|^2 + \|\delta_1 - \delta_0\|^2 \right) \\ \leq& \frac{\varsigma \alpha^\omega}{1-\alpha} \left(\|\beta_1 - \beta_0\|^2 + \|\zeta_1 - \zeta_0\|^2 + \|\delta_1 - \delta_0\|^2 \right), \end{aligned}$$

this implies

$$\lim_{\omega\to\infty} \left(\|\beta_\omega - \beta_{\omega+\varsigma}\|^2 + \|\zeta_\omega - \zeta_{\omega+\varsigma}\|^2 + \|\delta_\omega - \delta_{\omega+\varsigma}\|^2 \right) = 0,$$

by applying the fact $\alpha \in (0,1)$. Hence,

$$\lim_{\omega\to\infty} \|\beta_\omega - \beta_{\omega+\varsigma}\|^2 = 0,$$

$$\lim_{\omega\to\infty} \|\zeta_\omega - \zeta_{\omega+\varsigma}\|^2 = 0,$$

and

$$\lim_{\omega\to\infty} \|\delta_\omega - \delta_{\omega+\varsigma}\|^2 = 0.$$

Where $\eth$ is closed, there is $\beta, \zeta, \delta \in \eth$ such that $\{\beta_\omega\} \to \beta$, $\{\zeta_\omega\} \to \zeta$ and $\{\delta_\omega\} \to \delta$.

Now, by applying the continuity of Γ and Ξ, we obtain

$$\begin{aligned} \beta &= \lim_{\omega\to\infty} \beta_{\omega+1} = \lim_{\omega\to\infty} \Gamma(\beta_\omega, \zeta_\omega, \delta_\omega) = \Gamma(\beta, \zeta, \delta) \\ &= \lim_{\omega\to\infty} \Xi(\beta_{\omega+1}, \zeta_{\omega+1}, \delta_{\omega+1}) = \Xi(\beta, \zeta, \delta), \end{aligned}$$

also

$$\begin{aligned} \zeta &= \lim_{\omega\to\infty} \zeta_{\omega+1} = \lim_{\omega\to\infty} \Gamma(\zeta_\omega, \delta_\omega, \beta_\omega) = \Gamma(\zeta, \delta, \beta) \\ &= \lim_{\omega\to\infty} \Xi(\zeta_\omega, \delta_\omega, \beta_\omega) = \Xi(\zeta, \delta, \beta), \end{aligned}$$

and

$$\begin{aligned} \delta &= \lim_{\omega\to\infty} \delta_{\omega+1} = \lim_{\omega\to\infty} \Gamma(\delta_\omega, \beta_\omega, \zeta_\omega) = \Gamma(\delta, \beta, \zeta) \\ &= \lim_{\omega\to\infty} \Xi(\delta_{\omega+1}, \beta_{\omega+1}, \zeta_{\omega+1}) = \Xi(\delta, \beta, \zeta). \end{aligned}$$

This proves that (β, ζ, δ) is a common TFP of Γ and Ξ.

Now, to prove the uniqueness, we give the following corollary:

Corollary 1. *Let $\eth$ be a closed subset of a HS Ω, and $\Gamma : \eth^3 \to \eth$ be a continuous mapping which fulfills the $\varphi-$contraction. Then, Γ has a unique TFP in $\eth^3$.*

*Proof.*For uniqueness, suppose that $(\beta,\zeta,\delta)\in \eth^3$ is another TFP of Γ such that $\left(\widehat{\beta},\widehat{\zeta},\widehat{\delta}\right)\neq(\beta,\zeta,\delta)$. Consider

$$\begin{aligned}
&\left\|\beta-\widehat{\beta}\right\|^2\\
&\left\|\Gamma(\beta,\zeta,\delta)-\Gamma\left(\widehat{\beta},\widehat{\zeta},\widehat{\delta}\right)\right\|^2\\
&\le\varphi\left(\frac{\left\|\beta-\widehat{\beta}\right\|^2+\left\|\zeta-\widehat{\zeta}\right\|^2+\left\|\delta-\widehat{\delta}\right\|^2}{3},\right.\\
&\frac{\|\beta-\Gamma(\beta,\zeta,\delta)\|^2+\left\|\widehat{\beta}-\Gamma\left(\widehat{\beta},\widehat{\zeta},\widehat{\delta}\right)\right\|^2}{3},\\
&\frac{\|\zeta-\Gamma(\zeta,\delta,\beta)\|^2+\left\|\widehat{\zeta}-\Gamma\left(\widehat{\zeta},\widehat{\delta},\widehat{\beta}\right)\right\|^2}{3},\\
&\frac{\|\delta-\Gamma(\delta,\beta,\zeta)\|^2+\left\|\widehat{\delta}-\Gamma\left(\widehat{\delta},\widehat{\beta},\widehat{\zeta}\right)\right\|^2}{3},\\
&\frac{\left\|\widehat{\zeta}-\Gamma(\zeta,\delta,\beta)\right\|^2+\left\|\zeta-\Gamma\left(\widehat{\zeta},\widehat{\delta},\widehat{\beta}\right)\right\|^2}{3},\\
&\left.\frac{\left\|\widehat{\delta}-\Gamma(\delta,\beta,\zeta)\right\|^2+\left\|\delta-\Gamma\left(\widehat{\delta},\widehat{\beta},\widehat{\zeta}\right)\right\|^2}{3}\right),\\
&=\varphi\left(\frac{\left\|\beta-\widehat{\beta}\right\|^2+\left\|\zeta-\widehat{\zeta}\right\|^2+\left\|\delta-\widehat{\delta}\right\|^2}{3},\right.\\
&\left.0,0,0,\left\|\zeta-\widehat{\zeta}\right\|^2,\left\|\delta-\widehat{\delta}\right\|^2\right).
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
\left\|\zeta-\widehat{\zeta}\right\|^2\le\varphi&\left(\frac{\left\|\beta-\widehat{\beta}\right\|^2+\left\|\zeta-\widehat{\zeta}\right\|^2+\left\|\delta-\widehat{\delta}\right\|^2}{3},\right.\\
&\left.0,0,0,\left\|\delta-\widehat{\delta}\right\|^2,\left\|\beta-\widehat{\beta}\right\|^2\right).
\end{aligned}$$

Again, similarly

$$\begin{aligned}
\left\|\delta-\widehat{\delta}\right\|^2\le\varphi&\left(\frac{\left\|\beta-\widehat{\beta}\right\|^2+\left\|\zeta-\widehat{\zeta}\right\|^2+\left\|\delta-\widehat{\delta}\right\|^2}{3},\right.\\
&\left.0,0,0,\left\|\beta-\widehat{\beta}\right\|^2,\left\|\zeta-\widehat{\zeta}\right\|^2\right).
\end{aligned}$$

Using (ii) as $\varphi\in\Phi$, we have

$$\begin{aligned}
&\left\|\beta-\widehat{\beta}\right\|^2+\left\|\zeta-\widehat{\zeta}\right\|^2+\left\|\delta-\widehat{\delta}\right\|^2\\
&\le\alpha\left(\left\|\beta-\widehat{\beta}\right\|^2+\left\|\zeta-\widehat{\zeta}\right\|^2+\left\|\delta-\widehat{\delta}\right\|^2\right),
\end{aligned}$$

this leads to

$$\left\|\beta-\widehat{\beta}\right\|^2+\left\|\zeta-\widehat{\zeta}\right\|^2+\left\|\delta-\widehat{\delta}\right\|^2=0,$$

as $\alpha\in(0,1)$. Hence, $\left\|\beta-\widehat{\beta}\right\|^2=0$, $\left\|\zeta-\widehat{\zeta}\right\|^2=0$, and $\left\|\delta-\widehat{\delta}\right\|^2=0$, this implies that, $\beta=\widehat{\beta}$, $\zeta=\widehat{\zeta}$, and $\delta=\widehat{\delta}$, which contradicts our assumption. Thus, (β,ζ,δ) is a unique TFP of Γ.

*Example 2.*Let $\eth=[0,\frac{1}{2}]$ be a closed subset of a HS $\mathbb{R}$. Define $\Gamma:\eth^3\to\eth$ such that

$$\Gamma(\beta,\zeta,\delta)=\begin{cases}\frac{\beta-(\zeta+\delta)}{3}, & \beta\ge\zeta+\delta\\ 0, & \text{otherwise.}\end{cases}$$

Also, we define $\varphi:\mathbb{R}^6\to\mathbb{R}^+$ such that

$$\begin{aligned}
&\varphi(\beta,\zeta,\delta,\mu_1,\mu_2,\mu_3)\\
&=\beta+\max\{\zeta,\delta,\mu_1,\mu_2,\mu_3\}.
\end{aligned}$$

It is easy to observe that if $\beta<\zeta+\delta$ then $\varphi-$contraction is trivially fulfilled by Γ.

Consider $\beta\ge\zeta+\delta$, hence

$$\begin{aligned}
&\varphi\left(\frac{\|\beta-\beta^*\|^2+\|\zeta-\zeta^*\|^2+\|\delta-\delta^*\|^2}{3},\right.\\
&\frac{\|\beta-\Gamma(\beta,\zeta,\delta)\|^2+\|\beta^*-\Gamma(\beta^*,\zeta^*,\delta^*)\|^2}{3},\\
&\frac{\|\zeta-\Gamma(\zeta,\delta,\beta)\|^2+\|\zeta^*-\Gamma(\zeta^*,\delta^*,\beta^*)\|^2}{3},\\
&\frac{\|\delta-\Gamma(\delta,\beta,\zeta)\|^2+\|\delta^*-\Gamma(\delta^*,\beta^*,\zeta^*)\|^2}{3},\\
&\frac{\|\zeta^*-\Gamma(\zeta,\delta,\beta)\|^2+\|\zeta-\Gamma(\zeta^*,\delta^*,\beta^*)\|^2}{3},\\
&\left.\frac{\|\delta^*-\Gamma(\delta,\beta,\zeta)\|^2+\|\delta-\Gamma(\delta^*,\beta^*,\zeta^*)\|^2}{3}\right),\\
&=\varphi\left(\frac{\|\beta-\beta^*\|^2+\|\zeta-\zeta^*\|^2+\|\delta-\delta^*\|^2}{3},\right.\\
&\frac{\|2\beta+\zeta+\delta\|^2+\|2\beta^*+\zeta^*+\delta^*\|^2}{9},\\
&\frac{\|2\zeta+\delta+\beta\|^2+\|2\zeta^*+\delta^*+\beta^*\|^2}{9},\\
&\frac{\|2\delta+\beta+\zeta\|^2+\|2\delta^*+\beta^*+\zeta^*\|^2}{9},\\
&\frac{\|3\zeta^*-(\zeta-(\delta+\beta))\|^2+\|3\zeta-(\zeta^*-(\delta^*+\beta^*))\|^2}{9},\\
&\left.\frac{\|3\delta^*-(\delta-(\beta+\zeta))\|^2+\|3\delta-(\delta^*-(\beta^*+\zeta^*))\|^2}{9}\right),
\end{aligned}$$

$$= \frac{\|\beta-\beta^*\|^2+\|\zeta-\zeta^*\|^2+\|\delta-\delta^*\|^2}{3} + \max\left\{ \frac{\|2\beta+\zeta+\delta\|^2+\|2\beta^*+\zeta^*+\delta^*\|^2}{9}, \frac{\|2\zeta+\delta+\beta\|^2+\|2\zeta^*+\delta^*+\beta^*\|^2}{9}, \frac{\|2\delta+\beta+\zeta\|^2+\|2\delta^*+\beta^*+\zeta^*\|^2}{9}, \frac{\|3\zeta^*-(\zeta-(\delta+\beta))\|^2+\|3\zeta-(\zeta^*-(\delta^*+\beta^*))\|^2}{9}, \frac{\|3\delta^*-(\delta-(\beta+\zeta))\|^2+\|3\delta-(\delta^*-(\beta^*+\zeta^*))\|^2}{9} \right\},$$

$$\geq \frac{\|\beta-\beta^*\|^2+\|\zeta-\zeta^*\|^2+\|\delta-\delta^*\|^2}{3}$$
$$= \frac{\|\beta-\beta^*\|^2+\|\zeta^*-\zeta\|^2+\|\delta^*-\delta\|^2}{3}$$
$$\geq \frac{\|(\beta-\beta^*)+(\zeta^*-\zeta)+(\delta^*-\delta)\|^2}{9}$$
$$= \frac{\|(\beta-(\zeta+\delta))-(\beta^*-(\zeta^*+\delta^*))\|^2}{9}$$
$$= \left\|\Gamma(\beta,\zeta,\delta)-\Gamma\left(\widehat{\beta},\widehat{\zeta},\widehat{\delta}\right)\right\|^2.$$

Here, we used the fact of the inequality holds with both possible choices of the maximum value of the above-mentioned function. Then, all the postulates of Theorem 1 are fulfilled, proving that $(0,0,0)$ is a TFP of Γ.

*Remark.*If $\mu_1=\mu_2=\mu_3$ and $\beta=\zeta=\delta$, then the defined tripled implicit relation in Definition 8 would be given as below:

Assume that $\varphi:\mathbb{R}^6\to\mathbb{R}^+$ is a continuous function and it is non-decreasing in the fifth and the sixth argument; then, it verifies implicit relation for all $\beta,\mu_1>0$, that is, if

(i)

$$\beta \leq \varphi\left(\mu_1, \frac{\beta+\mu_1}{3}, \frac{\beta+\mu_1}{3}, \frac{\beta+\mu_1}{3}, \beta+\mu_1, \beta+\mu_1\right),$$

or

(ii)

$$\beta \leq \varphi(\mu_1,0,0,0,\mu_1,\mu_1).$$

Then, there exists a real number $\alpha\in(0,1)$ such that $\beta\leq\alpha(\mu_1)$.

Theorem 2.*Suppose that $\eth$ is a closed subset of a HS Ω, and $\Gamma:\eth^3\to\eth$ is a $\varphi-$contraction, then $(\{\beta_\omega,\zeta_\omega,\delta_\omega\})$ is a tripled asymptotically $\Gamma-$regular and Γ has a unique TFP in Ω^3 if and only if Γ is continuous at its TFP.*

*Proof.*Let (β,ζ,δ) be a TFP of Γ, this implies

$$\beta=\Gamma(\beta,\zeta,\delta),$$
$$\zeta=\Gamma(\zeta,\delta,\beta),$$

and

$$\delta=\Gamma(\delta,\beta,\zeta).$$

Consider three sequences $\{\beta_\omega\},\{\zeta_\omega\},\{\delta_\omega\}\in\eth$, such that $\beta_\omega\to\beta$, $\zeta_\omega\to\zeta$, $\delta_\omega\to\delta$, and the triple $(\{\beta_\omega\},\{\zeta_\omega\},\{\delta_\omega\})$ is asymptotically regular of Γ, this leads to

$$\lim_{\omega\to\infty}\|\beta_\omega-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|=0,$$
$$\lim_{\omega\to\infty}\|\zeta_\omega-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|=0,$$

and

$$\lim_{\omega\to\infty}\|\delta_\omega-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|=0.$$

Thus,

$$\|\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\Gamma(\beta,\zeta,\delta)\|^2 \leq \varphi\left(\frac{\|\beta_\omega-\beta\|^2+\|\zeta_\omega-\zeta\|^2+\|\delta_\omega-\delta\|^2}{3}, \frac{\|\beta_\omega-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2+\|\beta-\Gamma(\beta,\zeta,\delta)\|^2}{3}, \frac{\|\zeta_\omega-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2+\|\zeta-\Gamma(\zeta,\delta,\beta)\|^2}{3}, \frac{\|\delta_\omega-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2+\|\delta-\Gamma(\delta,\beta,\zeta)\|^2}{3}, \frac{\|\zeta-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2+\|\zeta_\omega-\Gamma(\zeta,\delta,\beta)\|^2}{3}, \frac{\|\delta-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2+\|\delta_\omega-\Gamma(\delta,\beta,\zeta)\|^2}{3} \right)$$

$$= \varphi\left(\frac{1}{3}\begin{pmatrix} \|\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\Gamma(\beta,\zeta,\delta)\|^2 \\ +\|\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)-\Gamma(\zeta,\delta,\beta)\|^2 \\ +\|\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)-\Gamma(\delta,\beta,\zeta)\|^2 \end{pmatrix}, 0,0,0, \frac{1}{3}\begin{pmatrix} \|\Gamma(\zeta,\delta,\beta)-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2 \\ +\|\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)-\Gamma(\zeta,\delta,\beta)\|^2 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} \|\Gamma(\delta,\beta,\zeta)-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2 \\ +\|\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)-\Gamma(\delta,\beta,\zeta)\|^2 \end{pmatrix} \right)$$

$$= \varphi\left(\frac{1}{3}\begin{pmatrix} \|\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\Gamma(\beta,\zeta,\delta)\|^2 \\ +\|\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)-\Gamma(\zeta,\delta,\beta)\|^2 \\ +\|\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)-\Gamma(\delta,\beta,\zeta)\|^2 \end{pmatrix}, 0,0,0, \|\Gamma(\zeta,\delta,\beta)-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2, \|\Gamma(\delta,\beta,\zeta)-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2 \right).$$

Similarly, we have

$$\begin{aligned}&\|\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)-\Gamma(\zeta,\delta,\beta)\|^2\\ &\le \varphi\left(\frac{1}{3}\begin{pmatrix}\|\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)-\Gamma(\zeta,\delta,\beta)\|^2\\ +\|\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)-\Gamma(\delta,\beta,\zeta)\|^2\\ +\|\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\Gamma(\beta,\zeta,\delta)\|^2\end{pmatrix},\right.\\ &0,0,0,\\ &\|\Gamma(\delta,\beta,\zeta)-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2,\\ &\left.\|\Gamma(\beta,\zeta,\delta)-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2\right).\end{aligned}$$

and

$$\begin{aligned}&\|\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)-\Gamma(\delta,\beta,\zeta)\|^2\\ &\le \varphi\left(\frac{1}{3}\begin{pmatrix}\|\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)-\Gamma(\delta,\beta,\zeta)\|^2\\ +\|\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\Gamma(\beta,\zeta,\delta)\|^2\\ +\|\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)-\Gamma(\zeta,\delta,\beta)\|^2\end{pmatrix},\right.\\ &0,0,0,\\ &\|\Gamma(\beta,\zeta,\delta)-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2,\\ &\left.\|\Gamma(\zeta,\delta,\beta)-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2\right).\end{aligned}$$

Now, applying hypothesis (ii) of Definition 8 of $\varphi\in\Phi-$family, we get

$$\begin{pmatrix}\|\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\Gamma(\beta,\zeta,\delta)\|^2\\ +\|\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)-\Gamma(\zeta,\delta,\beta)\|^2\\ +\|\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)-\Gamma(\delta,\beta,\zeta)\|^2\end{pmatrix}\le\alpha\begin{pmatrix}\|\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\Gamma(\beta,\zeta,\delta)\|^2\\ +\|\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)-\Gamma(\zeta,\delta,\beta)\|^2\\ +\|\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)-\Gamma(\delta,\beta,\zeta)\|^2\end{pmatrix}.$$

Using $\alpha\in(0,1)$ and passing $\omega\to\infty$,

$$\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\to\Gamma(\beta,\zeta,\delta),$$
$$\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\to\Gamma(\zeta,\delta,\beta)$$

and

$$\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\to\Gamma(\delta,\beta,\zeta).$$

Then, Γ is continuous at its TFP.

the other side, let Γ be continuous at $(\beta,\zeta,\delta)\in\eth^3$; then, from Theorem 1, Γ has a unique TFP. Now, Assume that $\{\beta_\omega\},\{\zeta_\omega\},\{\delta_\omega\}$ are three sequences such that $\beta_\omega\to\beta$, $\zeta_\omega\to\zeta$, $\delta_\omega\to\delta$, then as $\omega\to\infty$,

$$\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\to\Gamma(\beta,\zeta,\delta),$$
$$\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\to\Gamma(\zeta,\delta,\beta)$$

and

$$\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\to\Gamma(\delta,\beta,\zeta).$$

Also, we have

$$\|\beta_\omega-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2\to\|\beta-\Gamma(\beta,\zeta,\delta)\|^2=0,$$

$$\|\zeta_\omega-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2\to\|\zeta-\Gamma(\zeta,\delta,\beta)\|^2=0,$$

and

$$\|\delta_\omega-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2\to\|\delta-\Gamma(\delta,\beta,\zeta)\|^2=0.$$

This leads to

$$\lim_{\omega\to\infty}\|\beta_\omega-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|=0,$$

$$\lim_{\omega\to\infty}\|\zeta_\omega-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|=0,$$

and

$$\lim_{\omega\to\infty}\|\delta_\omega-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|=0.$$

4 Well-Posedness Theorem

In 1989, Blassi and Myjak [18] presented another approach of FPs, called well-posedness of a FP problem. The concept of well-posedness of a FP problem for a single-valued mapping can be observed in [18,20,21,22,23].

Now, we establish the well-posedness of a TFP problem of self-mapping in Corollary 1.

Definition 10. *Let Γ be a self-mapping on a HS Ω, then the FP problem of Γ is called a well-posed problem if*

(1) *Γ has a unique FP $\beta_0\in\Omega$,*

(2) for a sequence $\{\beta_\omega\}\in\Omega$ if

$$\lim_{\omega\to\infty}\|\beta_\omega-\Gamma(\beta_\omega)\|=0,$$

then

$$\lim_{\omega\to\infty}\|\beta_\omega-\beta_0\|=0.$$

Definition 11. *Assume that Ω is a HS and define $\Gamma:\Omega^3\to\Omega$. A FP problem on Ω^3 of a self-mapping Γ is called a well-posed problem if*

(1) *Γ has a unique FP $\beta_0\in\Omega$,*

(2) for asymptotically $\Gamma-$regular sequences $\{\beta_\omega\},\{\zeta_\omega\},\{\delta_\omega\}\in\Omega$

$$\widehat{\beta}=\lim_{\omega\to\infty}\beta_\omega,\ \widehat{\zeta}=\lim_{\omega\to\infty}\zeta_\omega,\ \widehat{\delta}=\lim_{\omega\to\infty}\delta_\omega,$$

where $\left(\widehat{\beta},\widehat{\zeta},\widehat{\delta}\right)$ is a TFP of Γ.

Theorem 3. *Let $\eth$ be a closed subset of a HS Ω, and $\Gamma,\Xi:\eth^3\to\eth$, such that*

(1) *Γ is continuous at its TFPs,*

(2) Γ is a $\varphi-$contraction,

(3)for any three sequences $\{\beta_\omega\},\{\zeta_\omega\},\{\delta_\omega\}$ and $\left(\widehat{\beta},\widehat{\zeta},\widehat{\delta}\right) \in T\left(\Gamma,\backepsilon^3\right)$, where T is the set of all common fixed points, we have

$$\lim_{\omega\to\infty}\|\beta_\omega-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\| = \lim_{\omega\to\infty}\|\zeta_\omega-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\| = \lim_{\omega\to\infty}\|\delta_\omega-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\| = 0.$$

Then, the TFP problem of Γ is well-posed.

Proof. Based on Corollary 1, Γ has a unique TFP, say $(\beta_0,\zeta_0,\delta_0)\in T\left(\Gamma,\backepsilon^3\right)$.

Suppose for the sequences $\{\beta_\omega\},\{\zeta_\omega\},\{\delta_\omega\}$, we have

$$\lim_{\omega\to\infty}\|\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\beta_\omega\| = \lim_{\omega\to\infty}\|\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)-\zeta_\omega\| = \lim_{\omega\to\infty}\|\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)-\delta_\omega\| = 0,$$

such that $(\beta_0,\zeta_0,\delta_0)\neq(\beta_\omega,\zeta_\omega,\delta_\omega)$ for any $\omega\in\mathbb{N}$. Using

$$\Gamma(\beta_0,\zeta_0,\delta_0)=\beta_0,\ \Gamma(\zeta_0,\delta_0,\beta_0)=\zeta_0,\ \Gamma(\delta_0,\beta_0,\zeta_0)=\delta_0,$$

we can write

$$\begin{aligned}
&\|\beta_0-\beta_\omega\|^2\\
&=\|\Gamma(\beta_0,\zeta_0,\delta_0)-\beta_\omega\|^2\\
&\le\|\Gamma(\beta_0,\zeta_0,\delta_0)-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2\\
&\quad+\|\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\beta_\omega\|^2\\
&\quad+2\langle\Gamma(\beta_0,\zeta_0,\delta_0)-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega),\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\beta_\omega\rangle\\
&\le\varphi\left(\frac{\|\beta_0-\beta_\omega\|^2+\|\zeta_0-\zeta_\omega\|^2+\|\delta_0-\delta_\omega\|^2}{3},\right.\\
&\frac{\|\beta_0-\Gamma(\beta_0,\zeta_0,\delta_0)\|^2+\|\beta_\omega-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\|^2}{3},\\
&\frac{\|\zeta_0-\Gamma(\zeta_0,\delta_0,\beta_0)\|^2+\|\zeta_\omega-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2}{3},\\
&\frac{\|\delta_0-\Gamma(\delta_0,\beta_0,\zeta_0)\|^2+\|\delta_\omega-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2}{3},\\
&\frac{\|\zeta_\omega-\Gamma(\zeta_0,\delta_0,\beta_0)\|^2+\|\zeta_0-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\|^2}{3},\\
&\left.\frac{\|\delta_\omega-\Gamma(\delta_0,\beta_0,\zeta_0)\|^2+\|\delta_0-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\|^2}{3}\right)\\
&\quad+\|\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\beta_\omega\|^2\\
&\quad+2\langle\Gamma(\beta_0,\zeta_0,\delta_0)-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega),\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)-\beta_\omega\rangle,
\end{aligned}$$

where

$$\lim_{\omega\to\infty}\|\beta_\omega-\Gamma(\beta_\omega,\zeta_\omega,\delta_\omega)\| = \lim_{\omega\to\infty}\|\zeta_\omega-\Gamma(\zeta_\omega,\delta_\omega,\beta_\omega)\| = \lim_{\omega\to\infty}\|\delta_\omega-\Gamma(\delta_\omega,\beta_\omega,\zeta_\omega)\| = 0,$$

for $(\beta_0,\zeta_0,\delta_0)\in T\left(\Gamma,\backepsilon^3\right)$, we obtain

$$\begin{aligned}
&\lim_{\omega\to\infty}\|\beta_0-\beta_\omega\|^2\\
&=\lim_{\omega\to\infty}\varphi\left(\frac{\|\beta_0-\beta_\omega\|^2+\|\zeta_0-\zeta_\omega\|^2+\|\delta_0-\delta_\omega\|^2}{3},\right.\\
&\left.0,0,0,\|\zeta_0-\zeta_\omega\|^2+\|\delta_0-\delta_\omega\|^2\right).
\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
&\lim_{\omega\to\infty}\|\zeta_0-\zeta_\omega\|^2\\
&=\lim_{\omega\to\infty}\varphi\left(\frac{\|\zeta_0-\zeta_\omega\|^2+\|\delta_0-\delta_\omega\|^2+\|\beta_0-\beta_\omega\|^2}{3},\right.\\
&\left.0,0,0,\|\delta_0-\delta_\omega\|^2+\|\beta_0-\beta_\omega\|^2\right),
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\omega\to\infty}\|\delta_0-\delta_\omega\|^2\\
&=\lim_{\omega\to\infty}\varphi\left(\frac{\|\delta_0-\delta_\omega\|^2+\|\beta_0-\beta_\omega\|^2+\|\zeta_0-\zeta_\omega\|^2}{3},\right.\\
&\left.0,0,0,\|\beta_0-\beta_\omega\|^2++\|\zeta_0-\zeta_\omega\|^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{\omega\to\infty}\|\beta_0-\beta_\omega\|^2+\lim_{\omega\to\infty}\|\zeta_0-\zeta_\omega\|^2+\lim_{\omega\to\infty}\|\delta_0-\delta_\omega\|^2\\
&\le\alpha\left(\lim_{\omega\to\infty}\|\beta_0-\beta_\omega\|^2+\lim_{\omega\to\infty}\|\zeta_0-\zeta_\omega\|^2\right.\\
&\left.\lim_{\omega\to\infty}\|\delta_0-\delta_\omega\|^2\right).
\end{aligned}$$

Then,

$$\lim_{\omega\to\infty}\beta_\omega=\beta_0,\ \lim_{\omega\to\infty}\zeta_\omega=\zeta_0 \text{ and } \lim_{\omega\to\infty}\delta_\omega=\delta_0,$$

which finishes the proof.

5 Conclusions and future work

In this work, we presented some important results for the existence and uniqueness for TFPs of self-mappings in a Hilbert space. Also, A well-posedness for tripled problems using generalized contraction conditions endowed with implicit relation is discussed. Furthermore, we gave an example to support our results. We established functions justifying an implicit relation from $\mathbb{R}^6$ to $\mathbb{R}^+$ in Definition 8. In the future, one can expand such functions justifying an implicit relation from $\mathbb{R}^n$ to $\mathbb{R}^+$, where $n\ge 7$.

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