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α-Cut Approach for Solving Fuzzy Rough Multi-Objective Quadratic Programming Problem

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Abstract: In this paper, we introduce a new operation on alpha cut for fuzzy rough and fully fuzzy rough multi-objective quadratic programming problems. Realistic quadratic programming problems often encounter uncertainty as well as indecision due to various factors that cannot be controlled. To overcome these limitations, fully fuzzy and fuzzy rough approaches are applied to such a problem. This paper proposes an effective method to solve the problem of fully fuzzy and fuzzy rough multi-objective quadratic programming where all the variables and parameters are fully fuzzy rough and fuzzy rough triangular numbers. Firstly, a fuzzy rough multi-objective quadratic problem has turned into an equivalent rough multi-objective quadratic programming problem with α-cut. Moreover, from the problem obtained, four crisp multi-objective quadratic programming problems are generated, and the resulting problems are solved as a crisp quadratic programming problem using a weighted method. We use Kuhn-Tucker conditions to solve the four crisp quadratic programming problems. An algorithm to solve (FRMOQP) problem with α-cut will be introduced. An illustrative example will be given. Secondly, a fully fuzzy rough multi-objective quadratic problem has turned into an equivalent fully rough multi-objective quadratic programming problem. Moreover, from the problem obtained four crisp multi-objective quadratic programming problems are generated, and the resulting problems are solved as a crisp quadratic programming problem using a weighted method. An algorithm to solve problems will be introduced. Finally, the effectiveness of the proposed procedure is demonstrated by numerical examples.
Keywords: Fuzzy Rough Intervals; α-Cut Approach, Multi-objective quadratic Programming, rough programming; Fully Rough programming

1. Introduction

Quadratic programming problems are applied in an increasing variety of practical fields, including applications in scheduling, planning and flow computations, inventory management [1], game theory, engineering modeling, design and control, problems involving economies of scale, facility allocation, and location problems, problems in microeconomics among others. Several applications portfolio selection [2] engineering design [3] molecular study [4], they may be used to solve some interesting combinatorial optimization problems, economics [5] and test problems for quadratic programming can be found in [6 -7]. So quadratic programming is one of the most important optimization techniques in operations research.

As ambiguity and vagueness are natural and ever-present in real-life situations requiring solutions, it makes perfect sense to attempt to address those using fuzzy and fully fuzzy quadratic and nonlinear programming problems. [8-14] which introduced some sufficient conditions were proposed to determine the optimal solution to the problem without solving it, directly. [15] Show an extension of Tanaka's method to solve fuzzy quadratic programming problems. In Kerre's method [16], a maximum of two LR fuzzy numbers is computed using the extension principle of Bellman and Zadeh [17,18] Which this article the objective function is crisp quadratic with fuzzy inequality constraints. It has presented an operative and novel method for solving problems which is carried out by performing two phases. [19] Based on the kuhu –tucker (KKT) necessary conditions that transform it to fuzzy linear programming then used dual simplex method. [20] All coefficients of this problem are characterized by L-R fuzzy numbers, the FFQP problem is converted into fully fuzzy linear programming using the Taylor series and hence into linear programming with an arbitrary initial point. [21] Uses a genetic algorithm with an acceptable membership degree which is desired by the DM, and through the human-computer interaction, the solutions preferred by the DM under different types of criteria can be achieved. [22] Solve the problem without changing it to a crisp program. Using Modified Kerre's method.

Also, for uncertainty, we used a rough set. the rough set theory, introduced by Pawlak [23,24], as a formal tool for dealing with imprecision and uncertainty in data [25] analysis the fundamental concept in rough set theory is to recognize every object in the universe based on information (data and knowledge). In a rough set, there are some objects, which cannot be characterized as the member of the set or the complement of the set, with certainty. Hence, the representation of a rough set can be approximated by a pair of two crisp sets, commonly known as lower and upper approximations. These approximation sets are commonly generated due to the existence of indiscernible (equivalence) relations. Many examples of applications of the rough set method to process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis, and other fields can be found in [26-30]. A fuzzy rough set [31] is an extension of the rough set. The concept of fuzzy rough sets, introduced by Dubois and Prade [32], plays an important role. In their study, the authors combined the fuzzy set and rough set to fulfill two different objectives. In fuzzy rough, replace the equivalence relation of the rough set with the fuzzy similarity relation. Since then, some important contributions to the hybridization of fuzzy and rough sets have been observed both in theory [33-37]) and in applications [38 -45] domains. The previous articles did not use the approach of operations using α-cut. Other work [46-50].

We propose a new approach for α-Cut Fuzzy rough and fully fuzzy rough multi-objective quadratic programming problems. We solve the $(FRMOQP)_\alpha$ and $(FFRMOQP)_\alpha$. Thus; the paper is organized as follows: Sect.2 shows some basic definitions of fuzzy theory, fuzzy number, and triangular fuzzy number. Short description of basic concepts of fuzzy rough numbers and the α-cut of fuzzy rough interval. In Sect.3 the α-Cut Approach for (FRMOQP) Problem and numerical example. In Sect.4 the α-Cut Approach for (FFRMOQP) Problem and numerical example. Finally, conclusions are presented in Sect. 5.

2. Preliminary

In this section, we give some basic notations and preliminary results which are essential tools for describing our main results. For details, we refer to [51-53].

Definition 2.1. Let X denote a universal set. Then a fuzzy subset A of X is defined by its membership function $\mu_{\tilde{A}}: X \to [0,1]$; which assigns a real number $\mu_A(x)$ in the interval [0, 1], to each element $x \in X$, where the value of $\mu_{\tilde{A}}(x)$ at x shows the grade of membership of x in A.

A fuzzy subset A can be characterized as a set of ordered pairs of element x and grade $\mu_A(x)$ and is often written

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), x \in X\}.$$

Definition 2.2. A fuzzy number $\tilde{A} = (a^\ell, a^m, a^u)$ is said to be a triangular fuzzy number if its membership function is given.

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{(x - a^\ell)}{(a^m - a^\ell)}, & a^\ell \le x \le a^m, \\ \dfrac{(a^u - x)}{(a^m - a^u)}, & a^m \le x \le a^u, \\ 0, & \text{otherewise}. \end{cases}$$

Definition 2.3. A triangular fuzzy number $\tilde{A} = (a^\ell, a^m, a^u)$ is said to be nonnegative fuzzy number if and only if $a^\ell \ge 0$. the set of non-negative fuzzy numbers may be represented by $F(R^+)$.

Definition 2.4. A triangular fuzzy number $\tilde{A} = (a^\ell, a^m, a^u)$ is said to be unrestricted fuzzy number if $a^\ell, a^m, a^u \in R$. The set of unrestricted triangular fuzzy numbers can be represented by F(R).

Definition 2.5. Let $\tilde{A} = (a^\ell, a^m, a^u), \tilde{B} = (b^\ell, b^m, b^u)$ be two triangular fuzzy numbers then.

$$(i) \tilde{A} \oplus \tilde{B} = (a^\ell, a^m, a^u) \oplus (b^\ell, b^m, b^u) = (a^\ell + b^\ell, a^m + b^m, a^u + b^u),$$

$$(ii) \tilde{A} - \tilde{B} = (a^\ell, a^m, a^u) - (b^\ell, b^m, b^u) = (a^\ell - b^u, a^m - b^m, a^u - b^\ell),$$

$$(iii)\ \tilde{A} \otimes \tilde{B} = (min(\varsigma), a^m b^m, max(\varsigma))$$
$$where, \varsigma = \{a^\ell b^\ell, a^\ell b^u, a^u b^\ell, a^u b^u\}.$$

$$(iv)\ k\tilde{A} = k(a^\ell, a^m, a^u) = (k a^\ell, k a^m, k a^u) \text{ if } k > 0$$

$$(v)\ k\tilde{A} = k(a^\ell, a^m, a^u) = (k a^u, k a^m, k a^\ell) \text{ if } k < 0$$

$$(vi) \tilde{A}/\tilde{B} = (a^\ell, a^m, a^u) \div (b^\ell, b^m, b^u) = (a^\ell \div b^u, a^m \div b^m, a^u \div b^\ell).$$

Definition2.6. Two triangular fuzzy numbers

$\tilde{A} = (a^\ell, a^m, a^u), \tilde{B} = (b^\ell, b^m, b^u)$ are said to be equal if and only if $a^\ell = b^\ell, a^m = b^m$ and $a^u = b^u$.

Definition 2.7. A ranking function is a function $R: F(R) \to R$, This maps each fuzzy number into the real line, where a natural order exists. Let $\tilde{A} = (a^\ell, a^m, a^u)$ be any triangular Fuzzy number, then $R(\tilde{A}) = \frac{1}{4}(a^\ell + 2a^m + a^u)$.

Definition 2.8. Let $\tilde{A} = (a^\ell, a^m, a^u), \tilde{B} = (b^\ell, b^m, b^u)$ be two triangular fuzzy numbers, then

$(i) A \le B$ iff $R(A) \le R(B)$,

Definition 2.9. Let X be denoting a compact set of real numbers. A fuzzy rough interval $\tilde{X}^R$ is defined as

$\tilde{X}^R = [\tilde{x}^{LAI} : \tilde{x}^{UAI}]$ where $\tilde{x}^{LAI}$ and $\tilde{x}^{UAI}$ Are fuzzy set called lower and upper approximation fuzzy numbers of $\tilde{X}^R$ with $\tilde{x}^{LAI} \subseteq \tilde{x}^{UAI}$.

Definition.2.10. A fuzzy rough interval $\tilde{A}^R = [\tilde{A}^{LAI} : \tilde{A}^{UAI}]$ is said to be normalized if $\tilde{A}^{LAI}$ and $\tilde{A}^{UAI}$ are normal.

Definition 2.11. Let

$\tilde{A}^R = [\tilde{A}^{LAI}:\tilde{A}^{UAI}]$ and $\tilde{B}^R = [\tilde{B}^{LAI}:\tilde{B}^{UAI}]$

are two fuzzy rough intervals. We say

$\tilde{A}^R = \tilde{B}^R$ iff $\tilde{A}^{LAI} = \tilde{B}^{LAI}$ *and* $\tilde{A}^{UAI} = \tilde{B}^{UAI}$.

Definition 2.12 [41] a fuzzy rough number $\tilde{A}^R$ is a triangular fuzzy rough number denoted by

$\tilde{A}^R = [(a^{\ell\ell}, a^m, a^{u\ell}):(a^{\ell u}, a^m, a^{uu})]$
where $a^{\ell\ell}, a^m, a^{u\ell}, a^{\ell u}$ *and* $a^{uu} \in R$
such that $a^{\ell u} \le a^{\ell\ell} \le a^m \le a^{u\ell} \le a^{uu}$
where $\tilde{A}^{LAI} = (a^{\ell\ell}, a^m, a^{u\ell}), \tilde{A}^{UAI} = (a^{\ell u}, a^m, a^{uu})$
and $\tilde{A}^{LAI} \subseteq \tilde{A}^{UAI}$.

The membership function can defined as

$$\mu_{\tilde{A}^R}(x) = \begin{cases} \mu_{\tilde{A}^{LAI}}(x) \begin{cases} \dfrac{x - a^{\ell\ell}}{a^m - a^{\ell\ell}}, & a^{\ell\ell} \le x \le a^m \\ \dfrac{a^{u\ell} - x}{a^{u\ell} - a^m}, & a^m \le x \le a^{u\ell} \\ 0, & \text{otherewise} \end{cases} \\ \mu_{\tilde{A}^{UAI}}(x) \begin{cases} \dfrac{x - a^{\ell u}}{a^m - a^{\ell u}}, & a^{\ell u} \le x \le a^m \\ \dfrac{a^{uu} - x}{a^{uu} - a^m}, & a^m \le x \le a^{uu} \\ 0, & \text{otherewise} \end{cases} \end{cases}$$

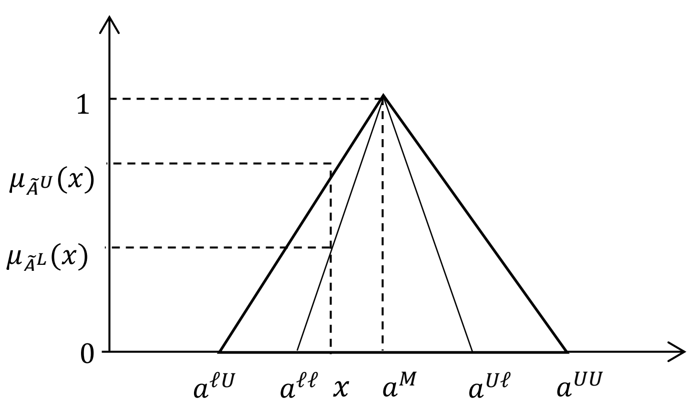


Figure 1

Note that:

$\tilde{A}^{LAI} = (a^{\ell\ell}, a^m, a^{u\ell}), \tilde{A}^{UAI} = (a^{\ell u}, a^m, a^{uu})$
and $\tilde{A}^{LAI} \subseteq \tilde{A}^{UAI}$. *where* $\mu_{\tilde{A}^{LAI}}(x)$ and $\mu_{\tilde{A}^{UAI}}(x)$

Are membership functions of lower and upper approximation triangular fuzzy number of $\mu_{\tilde{A}^R}(x)$.

Definition 2.13. [55] The α-cut of fuzzy rough interval $\tilde{A}^R$ is defined as:

$(\tilde{A}^R)_\alpha = [\tilde{A}^{LAI}_\alpha : \tilde{A}^{UAI}_\alpha]$

where A^{LAI}_α *and* A^{UAI}_α *are* intervals with $A^{LAI}_\alpha \subseteq_{LR} A^{UAI}_\alpha$.
$\tilde{A}^R = [(a^{\ell\ell}, a^m, a^{u\ell}):(a^{\ell u}, a^m, a^{uu})]$ as :

$(\tilde{A}^R)_\alpha = [a^{\ell\ell}(\alpha), a^{u\ell}(\alpha): a^{\ell u}(\alpha), a^{uu}(\alpha)]$,

Where

$\tilde{A}^{LAI}_\alpha = [a^{\ell\ell} + (a^m - a^{\ell\ell})\alpha, a^{u\ell} + (a^m - a^{u\ell})\alpha]$

And

$\tilde{A}^{UAI}_\alpha = [a^{\ell u} + (a^m - a^{\ell u})\alpha, a^{uu} + (a^m - a^{uu})\alpha]$

$, \alpha \epsilon [0,1]$

Definition 2.14. The arithmetic operations for TFRN

let $\tilde{A}^R \ge 0$ and $\tilde{B}^R \ge 0$

Be two fuzzy rough intervals, then

$(i) \tilde{A}^R \oplus \tilde{B}^R = [(\tilde{A}^{LAI} \oplus \tilde{B}^{LAI}):(\tilde{A}^{UAI} \oplus \tilde{B}^{UAI})]$.

$(ii)\ \tilde{A}^R - \tilde{B}^R = [(\tilde{A}^{LAI} - \tilde{B}^{LAI}):(\tilde{A}^{UAI} - \tilde{B}^{UAI})]$.

$(iii)\ \tilde{A}^R \otimes \tilde{B}^R = [(\tilde{A}^{LAI} \otimes \tilde{B}^{LAI}):(\tilde{A}^{UAI} \otimes \tilde{B}^{UAI})]$.

$(iv)\ \tilde{A}^R / \tilde{B}^R = [(\tilde{A}^{LAI} / \tilde{B}^{LAI}):(\tilde{A}^{UAI} / \tilde{B}^{UAI})]$.

3. α-Cut Approach for FRMOQP Problem

The multi-objective quadratic programming problems with fuzzy rough coefficients (FRMOQP) problem is defined as follows:

$$\left.\begin{array}{l} (FRMOQP): max\ f_r^R = \sum_{j=1}^{n} \tilde{c}^R_{r_j} x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{q}^R_{r_{ij}} x_i x_j \\ \qquad , r = 1,2,\dots,k \\ \text{Subject to} \quad \sum_{j=1}^{n} \tilde{a}^R_{ij} x_j \le \tilde{b}^R_i , \\ x_j \ge 0 , i \in I = \{1,2,\dots,n\}; \\ j \in J = \{1,2,\dots,n\} \end{array}\right\} \quad (1)$$

Where

$\tilde{c}^R_{r_j}, q^R_{r_{ij}}, a^R_{ij}, \tilde{b}^R_i, i \in I, j \in J, r = 1,2,\dots\dots,k$

Fuzzy rough coefficient. The problem (1) can be written as fuzzy rough interval as:

$$\left.\begin{array}{l} (FRMOQP): \max[\widetilde{QP}^{LAI}_r : \widetilde{QP}^{UAI}_r] \\ = \sum_{j=1}^{n} [\tilde{c}^{LAI}_{r_j} : \tilde{c}^{UAI}_{r_j}] x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} [\tilde{q}^{LAI}_{r_{ij}}, \tilde{q}^{UAI}_{r_{ij}}] x_i x_j \\ \qquad , r = 1,2,\dots,k \\ \text{Subject to} \quad \sum_{j=1}^{n} [\tilde{a}^{LAI}_{ij} : \tilde{a}^{UAI}_{ij}] x_j \le [\tilde{b}^{LAI}_i : \tilde{b}^{UAI}_i], \\ x_j \ge 0 , i \in I = \{1,2,\dots,n\}; \\ j \in J = \{1,2,\dots,n\} \end{array}\right\} \quad (2)$$

Such that $\tilde{c}^{LAI}_{r_j}, \tilde{c}^{UAI}_{r_j}, \tilde{q}^{LAI}_{r_{ij}}, \tilde{q}^{UAI}_{r_{ij}}, \tilde{a}^{LAI}_{ij}, \tilde{a}^{UAI}_{ij}, \tilde{b}^{LAI}_i, \tilde{b}^{UAI}_i$ are triangular fuzzy numbers

Where

$\tilde{c}_{rj}^{L} = (c_{rj}^{\ell\ell}, c_{rj}^{m}, c_{rj}^{u\ell})$, $\tilde{c}_{rj}^{U} = (c_{rj}^{\ell u}, c_{rj}^{m}, c_{rj}^{uu})$,
$\tilde{a}_{ij}^{L} = (a_{ij}^{\ell\ell}, a_{ij}^{m}, a_{ij}^{u\ell})$, $\tilde{a}_{ij}^{U} = (a_{ij}^{\ell u}, a_{ij}^{m}, a_{ij}^{uu})$,
$\tilde{b}_{i}^{L} = (b_{i}^{\ell\ell}, b_{i}^{m}, b_{i}^{u\ell})$, $\tilde{b}_{i}^{U} = (b_{i}^{\ell u}, b_{i}^{m}, b_{i}^{uu})$,
$\tilde{q}_{rij}^{L} = (q_{rij}^{\ell\ell}, q_{rij}^{m}, q_{rij}^{u\ell})$, $\tilde{q}_{rij}^{U} = (q_{rij}^{\ell u}, q_{rij}^{m}, q_{rij}^{uu})$.

According definition 2.13.we used α-cut (α-level set) of coefficient as:

$(\tilde{c}_{rj}^{L})_\alpha = ((c_{rj}^{\ell\ell})_\alpha, (c_{rj}^{u\ell})_\alpha)$, $(\tilde{c}_{rj}^{U})_\alpha = ((c_{rj}^{\ell u})_\alpha, (c_{rj}^{uu})_\alpha)$,
$(\tilde{c}_{rj}^{L})_\alpha = [c_{rj}^{\ell\ell} + (c_{rj}^{m} - c_{rj}^{\ell\ell})\,\alpha, c_{rj}^{u\ell} + (c_{rj}^{m} - c_{rj}^{u\ell})\alpha]$

$(\tilde{c}_{rj}^{U})_\alpha = [c_{rj}^{\ell u} + (c_{rj}^{m} - c_{rj}^{\ell u})\,\alpha, c_{rj}^{uu} + (c_{rj}^{m} - c_{rj}^{uu})\alpha]$

$(\tilde{a}_{ij}^{L})_\alpha = ((a_{ij}^{\ell\ell})_\alpha, (a_{ij}^{u\ell})_\alpha)$, $(\tilde{a}_{ij}^{U})_\alpha = ((a_{ij}^{\ell u})_\alpha, (a_{ij}^{uu})_\alpha)$,

$(\tilde{a}_{ij}^{L})_\alpha = [a_{ij}^{\ell\ell} + (a_{ij}^{m} - a_{ij}^{\ell\ell})\,\alpha, a_{ij}^{u\ell} + (a_{ij}^{m} - a_{ij}^{u\ell})\,\alpha]$

$(\tilde{a}_{ij}^{U})_\alpha = [a_{ij}^{\ell u} + (a_{ij}^{m} - a_{ij}^{\ell u})\,\alpha, a_{ij}^{uu} + (a_{ij}^{m} - a_{ij}^{uu})\,\alpha]$

$(\tilde{b}_{i}^{L})_\alpha = ((b_{i}^{\ell\ell})_\alpha, (b_{i}^{u\ell})_\alpha)$, $(\tilde{b}_{i}^{U})_\alpha = ((b_{i}^{\ell u})_\alpha, (b_{i}^{uu})_\alpha)$,
$(\tilde{b}_{i}^{L})_\alpha = [b_{i}^{\ell\ell} + (b_{i}^{m} - b_{i}^{\ell\ell})\,\alpha, b_{i}^{u\ell} + (b_{i}^{m} - b_{i}^{u\ell})\,\alpha]$

$(\tilde{b}_{i}^{U})_\alpha = [b_{i}^{\ell u} + (b_{i}^{m} - b_{i}^{\ell u})\,\alpha, b_{i}^{uu} + (b_{i}^{m} - b_{i}^{uu})\,\alpha]$

$(\tilde{q}_{rij}^{L})_\alpha = ((q_{rij}^{\ell\ell})_\alpha, (q_{rij}^{u\ell})_\alpha)$, $(\tilde{q}_{rij}^{U})_\alpha = ((q_{rij}^{\ell u})_\alpha, (q_{rij}^{uu})_\alpha)$.
$(\tilde{q}_{rij}^{L})_\alpha = [q_{rij}^{\ell\ell} + (q_{rij}^{m} - q_{rij}^{\ell\ell})\,\alpha, q_{rij}^{u\ell} + (q_{rij}^{m} - q_{rij}^{u\ell})\,\alpha]$
$(\tilde{q}_{rij}^{U})_\alpha = [q_{rij}^{\ell u} + (q_{rij}^{m} - q_{rij}^{\ell u})\,\alpha, q_{rij}^{uu} + (q_{rij}^{m} - q_{rij}^{uu})\,\alpha]$

To convert FRMOQPP to RMOQPP and get as:

$$\left.\begin{array}{l}(RMOQP_\alpha): max[\,\widetilde{QP}_{r\alpha}^{L}:\widetilde{QP}_{r\alpha}^{U}] \\ = \sum_{j=1}^{n} [(\tilde{c}_{r_j}^{L})_\alpha:(\tilde{c}_{r_j}^{U})_\alpha]x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[(\tilde{q}_{r_{ij}}^{L})_\alpha:(\tilde{q}_{r_{ij}}^{U})_\alpha]x_i x_j \\ \qquad ,r = 1,2,\ldots,k \\ \text{Subject to} \\ \sum_{j=1}^{n}[(\tilde{a}_{ij}^{L})_\alpha:(\tilde{a}_{ij}^{U})_\alpha]x_j \le [(\tilde{b}_{i}^{L})_\alpha:(\tilde{b}_{i}^{U})_\alpha], \\ x_j \ge 0\,, i \in I = \{1,2,\ldots,n\}; \\ j \in J = \{1,2,\ldots,n\}, \alpha \in (0,1]\end{array}\right\} (3)$$

We can write (3) as:

$$\left.\begin{array}{l}(FRMOQP_\alpha): max[\,\widetilde{QP}_{r\alpha}^{L}:\widetilde{QP}_{r\alpha}^{U}] \\ = \sum_{j=1}^{n} [((c_{r_j}^{\ell\ell})_\alpha, (c_{r_j}^{u\ell})_\alpha):((c_{r_j}^{\ell u})_\alpha, (c_{r_j}^{uu})_\alpha)]x_j + \\ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[((q_{r_{ij}}^{\ell\ell})_\alpha, (q_{r_{ij}}^{u\ell})_\alpha):((q_{r_{ij}}^{\ell u})_\alpha, (q_{r_{ij}}^{uu})_\alpha)]x_i x_j \\ \qquad ,r = 1,2,\ldots,k \\ \text{Subject to} \\ \sum_{j=1}^{n}[((a_{ij}^{\ell\ell})_\alpha, (a_{ij}^{u\ell})_\alpha)):((a_{ij}^{\ell u})_\alpha, (a_{ij}^{uu})_\alpha)]x_j \\ \quad \le [((b_{i}^{\ell\ell})_\alpha, (b_{i}^{u\ell})_\alpha):((b_{i}^{\ell u})_\alpha, (b_{i}^{uu})_\alpha)], \\ x_j \ge 0\,, i \in I = \{1,2,\ldots,n\}; \\ j \in J = \{1,2,\ldots,n\}, \alpha \in (0,1]\end{array}\right\} (4)$$

Where

$QP_{r\alpha}^{R}(x) = [[QP_{r\alpha}^{\ell\ell}(x), QP_{r\alpha}^{u\ell}(x)]:[QP_{r\alpha}^{\ell u}(x), QP_{r\alpha}^{uu}(x)]]$.

Therefore the problem (4) decomposes to a multi-objective quadratic programming problem defines as follows:

$$\left.\begin{array}{l}\max \mathrm{QP}_{r\alpha}^{uu}(x) = \sum_{j=1}^{n} (c_{r_j}^{uu})_\alpha x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathrm{q}_{r_{ij}}^{uu})_\alpha x_i x_j, r = 1, 2, \ldots, k \\ \text{Subject to} \\ \sum_{j=1}^{n}(\mathrm{a}_{ij}^{\ell u})_\alpha x_j \le (\mathrm{b}_i^{uu})_\alpha\,, \\ x_j \ge 0\,, i \in I = \{1, 2, \ldots, n\}; \\ j \in J = \{1, 2, \ldots, n\},\ \alpha \in (0,1]\end{array}\right\} (5)$$

$$\left.\begin{array}{l}\max \mathrm{QP}_{r\alpha}^{u\ell} = \sum_{j=1}^{n}(\mathrm{c}_{r_j}^{u\ell})_\alpha)x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathrm{q}_{r_{ij}}^{u\ell})_\alpha)x_i x_j, r = 1, 2, \ldots, k \\ \text{Subject to} \\ \sum_{j=1}^{n}(\mathrm{a}_{ij}^{\ell\ell})_\alpha x_j \le (\mathrm{b}_i^{u\ell})_\alpha), \\ x_j \ge 0\,, i \in I = \{1, 2, \ldots, n\}; \\ j \in J = \{1, 2, \ldots, n\}, \alpha \in (0,1]\end{array}\right\} (6)$$

$$\left.\begin{array}{l}\max \mathrm{QP}_{r\alpha}^{\ell\ell} = \sum_{j=1}^{n}(\mathrm{c}_{r_j}^{\ell\ell})_\alpha x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathrm{q}_{r_{ij}}^{\ell\ell})_\alpha x_i x_j, r = 1, 2, \ldots, k \\ \text{Subject to} \\ \sum_{j=1}^{n}(\mathrm{a}_{ij}^{u\ell})_\alpha)x_j \le (\mathrm{b}_i^{\ell\ell})_\alpha, \\ x_j \ge 0\,, i \in I = \{1, 2, \ldots, n\}; \\ j \in J = \{1, 2, \ldots, n\}, \alpha \in (0,1]\end{array}\right\} (7)$$

$$\left.\begin{array}{l}\max \mathrm{QP}_{r\alpha}^{\ell u} = \sum_{j=1}^{n}(\mathrm{c}_{r_j}^{\ell u})_\alpha x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathrm{q}_{r_{ij}}^{\ell u})_\alpha x_i x_j, r = 1, 2, \ldots, k \\ \text{Subject to} \\ \sum_{j=1}^{n}(a_{ij}^{uu})_\alpha x_j \le ((b_i^{\ell u})_\alpha, \\ x_j \ge 0\,, i \in I = \{1, 2, \ldots, n\}; \\ j \in J = \{1, 2, \ldots, n\}, \alpha \in (0,1]\end{array}\right\} (8)$$

Now, using the weighted sum method to convert the MOQP problem to quadratic programming problem:

$$\left.\begin{array}{l}\max \mathrm{QP}_{\alpha}^{uu}(\omega^*) = \sum_{r=1}^{k}\omega_r\left(\sum_{j=1}^{n}(c_{r_j}^{uu})_\alpha x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathrm{q}_{r_{ij}}^{uu})_\alpha x_i x_j\right) \\ \text{Subject to} \\ \sum_{j=1}^{n}(\mathrm{a}_{ij}^{\ell u})_\alpha x_j \le (b_i^{uu})_\alpha), \\ x_j \ge 0\,, i \in I = \{1, 2, \ldots, n\};\ j \in J = \{1, 2, \ldots, n\} \\ \alpha \in (0,1]\ \ , \omega_r \in w\end{array}\right\} (9)$$

$$\left.\begin{array}{l}\max \mathrm{Q}\,\mathrm{P}_{\alpha}^{u\ell}(\omega^*)=\sum_{r=1}^{k}\omega_r\left(\sum_{j=1}^{n}(c_{r_j}^{u\ell})_{\alpha})x_j+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{r_{ij}}^{u\ell})_{\alpha})x_i x_j\right)\\ \quad\text{Subject to}\\ \quad\sum_{j=1}^{n}(a_{ij}^{\ell\ell})_{\alpha}x_j\le(\mathrm{b}_i^{u\ell})_{\alpha},\\ \quad x_j\ge 0\ ,i\in I=\{1,2,...,n\};\ \ j\in J=\{1,2,...,n\}\\ \quad\alpha\ \in\ (0,1]\ ,\omega_r\in w\end{array}\right\}(10)$$

$$\left.\begin{array}{l}\max \mathrm{QP}_{\alpha}^{\ell\ell}(\omega^*)=\sum_{r=1}^{k}\omega_r\left(\sum_{j=1}^{n}(c_{r_j}^{\ell\ell})_{\alpha}x_j+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathrm{q}_{r_{ij}}^{\ell\ell})_{\alpha}x_i x_j\right)\\ \quad\text{Subject to}\\ \quad\sum_{j=1}^{n}(a_{ij}^{u\ell})_{\alpha})x_j\le(b_i^{\ell\ell})_{\alpha},\\ \quad x_j\ge 0\ ,i\in I=\{1,2,...,n\};\ \ j\in J=\{1,2,...,n\}\\ \quad\alpha\ \in\ (0,1]\ ,\omega_r\in w\end{array}\right\}(11)$$

$$\left.\begin{array}{l}\max \mathrm{QP}_{\alpha}^{\ell u}(\omega^*)=\sum_{r=1}^{k}\omega_r\left(\sum_{j=1}^{n}(\mathrm{c}_{r_j}^{\ell u})_{\alpha}x_j+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}((q_{r_{ij}}^{\ell u})_{\alpha}x_i x_j\right)\\ \quad\text{Subject to}\\ \quad\sum_{j=1}^{n}(a_{ij}^{uu})_{\alpha}x_j\le(b_i^{\ell u})_{\alpha},\\ \quad x_j\ge 0\ ,i\in I=\{1,2,...,n\};\ \ j\in J=\{1,2,...,n\}\\ \quad\alpha\ \in\ (0,1]\ ,\omega_r\in w\end{array}\right\}(12)$$

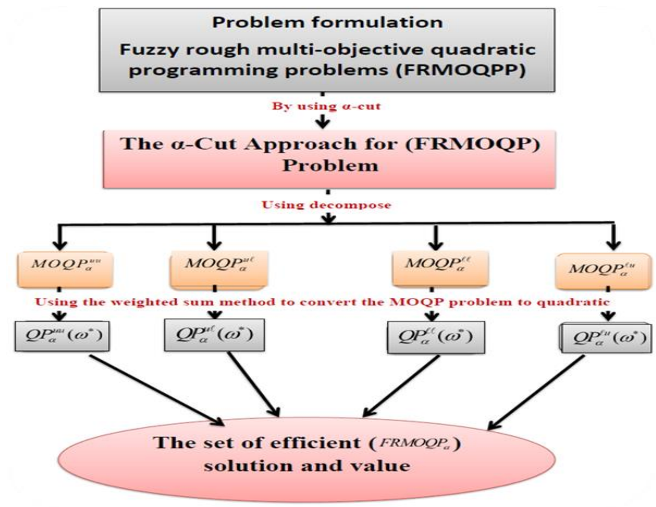
The optimal solutions of $QP_{\alpha}^{uu}(\omega^*), QP_{\alpha}^{u\ell}(\omega^*), QP_{\alpha}^{\ell\ell}(\omega^*)$ and $QP_{\alpha}^{\ell u}(\omega^*)$ Problems are the efficient solutions of $MOQP_{\alpha}^{uu}, MOQP_{\alpha}^{u\ell}, MOQP_{\alpha}^{\ell\ell}$ and $MOQP_{\alpha}^{\ell u}$ problems, respectively.

α-Cut Algorithm for FRMOQP problem

1) Convert the problem (1) to the form (2).
2) Use α-level cuts to deal with a fuzziness of fuzzy rough coefficients as a form (3).
3) Use decompose technique for problem (4)to get the MOQP problem $MOQP_{\alpha}^{uu}, MOQP_{\alpha}^{u\ell}, MOQP_{\alpha}^{\ell\ell}$ and $MOQP_{\alpha}^{\ell u}$
4) Use one of the secularization methods; say the weights method, to convert each problem $MOQP_{\alpha}^{uu}, MOQP_{\alpha}^{u\ell}, MOQP_{\alpha}^{\ell\ell}$ and $MOQP_{\alpha}^{\ell u}$ with a single objective in theorem $QP_{\alpha}^{uu}(\omega^*), QP_{\alpha}^{u\ell}(\omega^*), QP_{\alpha}^{\ell\ell}(\omega^*)$ and $QP_{\alpha}^{\ell u}(\omega^*)$

   For $\omega=\omega^*\in W$ find the optimal solution of each quadratic programming proble $QP_{\alpha}^{uu}(\omega^*), QP_{\alpha}^{u\ell}(\omega^*)$, $QP_{\alpha}^{\ell\ell}(\omega^*)$ and $QP_{\alpha}^{\ell u}(\omega^*)$.

A flowchart of the solution steps is provided for further illustration:



Example 1: Consider the following FRMOQP

$$\text{Max }\ MOQP_r^R(x)=$$

$$\left(\begin{array}{l}\frac{1}{2}(x_1\quad x_2)\otimes\begin{pmatrix}[[10,15,20]:[5,15,25]] & [[2,4,6]:[1,4,8]]\\ [[2,4,6]:[1,4,8]] & [[8,10,12]:[6,10,14]]\end{pmatrix}\otimes\begin{pmatrix}x_1\\ x_2\end{pmatrix},\\ \frac{1}{2}(x_1\quad x_2)\otimes\begin{pmatrix}[[6,8,14]:[4,8,16]] & [[6,20,25]:[5,20,28]]\\ [[6,20,25]:[5,20,28]] & [[12,14,20]:[10,14,22]]\end{pmatrix}\otimes\begin{pmatrix}x_1\\ x_2\end{pmatrix}\end{array}\right)$$

$$s.t.$$

$$\begin{pmatrix}[[4,8,16]:[2,8,20]] & [[6,12,18]:[3,12,28]]\\ [[6,12,18]:[3,12,28]] & [[10,20,36]:[8,20,60]]\end{pmatrix}\otimes\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\le\begin{pmatrix}[[60,70,110]:[55,70,120]]\\ [[110,120,140]:[30,120,190]]\end{pmatrix}$$

$$\max\ MOQP_r^R=$$

$$\begin{pmatrix}[[5,7.5,10]:[2.5,7.5,12.5]]\mathrm{x}_1^2+[[2,4,6]:[1,4,8]]\mathrm{x}_1x_2+[[4,5,6]:[3,5,7]]\mathrm{x}_2^2,\\ [[3,4,7]:[2.5,4,8]]\mathrm{x}_1^2+[[6,20,25]:[5,20,28]]\mathrm{x}_1x_2+[[6,7,10]:[5,7,11]]\mathrm{x}_2^2\end{pmatrix}$$

$$s.t.$$

$$[[4,8,16]:[2,8,20]]x_1+[[6,12,18]:[3,12,28]]x_2$$
$$\le\ [[60,70,110]:[55,70,120]]$$
$$[[6,12,18]:[3,12,28]]x_1+[[10,20,36]:[8,20,60]]x_2$$
$$\le\ [[110,120,140]:[30,120,190]]$$
$$\mathrm{x}_1\ge 0, x_2\ge 0.$$

Now we using α-cut for all triangular fuzzy number where

$$(\tilde{c}_{rj}^{L})_{\alpha}=[\mathrm{c}_{rj}^{\ell\ell}+(\mathrm{c}_{rj}^{m}-\mathrm{c}_{rj}^{\ell\ell})\alpha, c_{rj}^{u\ell}+(\mathrm{c}_{rj}^{m}-c_{rj}^{u\ell})\alpha]$$

$$(\tilde{c}_{rj}^{U})_{\alpha}=[\mathrm{c}_{rj}^{\ell u}+(\mathrm{c}_{rj}^{m}-\mathrm{c}_{rj}^{\ell u})\alpha, c_{rj}^{uu}+(c_{rj}^{m}-c_{rj}^{uu})\alpha]$$

$$(a_{ij}^{L})_{\alpha}=[\mathrm{a}_{ij}^{\ell\ell}+(\mathrm{a}_{ij}^{m}-\mathrm{a}_{ij}^{\ell\ell})\alpha, \mathrm{a}_{ij}^{u\ell}+(\mathrm{a}_{ij}^{m}-\mathrm{a}_{ij}^{u\ell})\alpha]$$

$$(a_{ij}^{U})_{\alpha}=[\mathrm{a}_{ij}^{\ell u}+(\mathrm{a}_{ij}^{m}-\mathrm{a}_{ij}^{\ell u})\alpha, \mathrm{a}_{ij}^{uu}+(\mathrm{a}_{ij}^{m}-\mathrm{a}_{ij}^{uu})\alpha]$$

$$(\tilde{b}_i^L)_\alpha = [b_i^{\ell\ell} + (b_i^m - b_i^{\ell\ell})\alpha, b_i^{u\ell} + (b_i^m - b_i^{u\ell})\alpha]$$
$$(\tilde{b}_i^U)_\alpha = [b_i^{\ell u} + (b_i^m - b_i^{\ell u})\alpha, b_i^{uu} + (b_i^m - b_i^{uu})\alpha]$$
$$(q_{rij}^L)_\alpha = [q_{rij}^{\ell\ell} + (q_{rij}^m - q_{rij}^{\ell\ell})\alpha, q_{rij}^{u\ell} + (q_{rij}^m - q_{rij}^{u\ell})\alpha]$$
$$(q_{rij}^U)_\alpha = [q_{rij}^{\ell u} + (q_{rij}^m - q_{rij}^{\ell u})\alpha, q_{rij}^{uu} + (q_{rij}^m - q_{rij}^{uu})\alpha]$$

We can now choose thus can be write $\max MOQP_r^R =$

$$\begin{pmatrix} [[6.25,8.75]:[5,10]]x_1^2 + [[3,5]:[2.5,6]]x_1x_2 + [[4.5,5.5]:[4,6]]x_2^2, \\ [[3.5,5.5]:[3.25,6]]x_1^2 + [[13,22.5]:[12.5,24]]x_1x_2 + [[6.5,8.5]:[6,9]]x_2^2 \end{pmatrix}$$

s.t.

$$[[6,12]:[5,14]]x_1 + [[9,15]:[7.5,20]]x_2 \le [[65,90]:[62.5,95]]$$
$$[[9,15]:[7.5,20]]x_1 + [[15,28]:[14,40]]x_2 \le [[115,130]:[75,155]]$$
$$x_1 \ge 0, x_2 \ge 0.$$

the above problem can be written as the following problems:

$$\max \ MOQP_{r\alpha}^{uu} = \begin{pmatrix} 10x_1^2 + 6x_1x_2 + 6x_2^2, \\ 6x_1^2 + 24x_1x_2 + 9x_2^2 \end{pmatrix}$$
$$s.t.$$
$$5x_1 + 7.5x_2 \le 95$$
$$7.5x_1 + 14x_2 \le 155$$
$$x_1 \ge 0, \ x_2 \ge 0$$

$$\max \ MOQP_{r\alpha}^{u\ell} = \begin{pmatrix} 8.75x_1^2 + 5x_1x_2 + 5.5x_2^2, \\ 5.5x_1^2 + 22.5x_1x_2 + 8.5x_2^2 \end{pmatrix}$$
$$s.t.$$
$$6x_1 + 9x_2 \le 90$$
$$9x_1 + 15x_2 \le 130$$
$$x_1 \ge 0, x_2 \ge 0$$

$$\max \ MOQP_{r\alpha}^{\ell\ell} = \begin{pmatrix} 6.25x_1^2 + 3x_1x_2 + 4.5x_2^2, \\ 3.5x_1^2 + 13x_1x_2 + 6.5x_2^2 \end{pmatrix}$$
$$s.t.$$
$$12x_1 + 15x_2 \le 65$$
$$15x_1 + 28x_2 \le 115$$
$$x_1 \ge 0, \ x_2 \ge 0$$

$$\max \ MOQP_{r\alpha}^{\ell u} = \begin{pmatrix} 5x_1^2 + 2.5x_1x_2 + 4x_2^2, \\ 3.25x_1^2 + 12.5x_1x_2 + 6x_2^2 \end{pmatrix}$$
$$s.t.$$
$$14x_1 + 20x_2 \le 62.5$$
$$20x_1 + 40x_2 \le 75$$
$$x_1 \ge 0, \ x_2 \ge 0$$

For $\omega_1 = \omega_2 = 0.5$ we getting the corresponded weighting quadratic programming problems $QP_\alpha^{uu}(\omega^*)$, $QP_\alpha^{u\ell}(\omega^*)$ $QP_\alpha^{\ell\ell}(\omega^*)$ and $QP_\alpha^{\ell u}(\omega^*)$, For $MOQP_{r\alpha}^{uu}, MOQP_{r\alpha}^{u\ell}, MOQP_{r\alpha}^{\ell\ell}$, and $MOQP_{r\alpha}^{\ell u}$ respectively, can be described as follows:

$$\max \ QP_\alpha^{uu}(\omega^*) = \left(8x_1^2 + 15x_1x_2 + 7.5x_2^2\right)$$
$$s.t.$$
$$5x_1 + 7.5x_2 \le 95$$
$$7.5x_1 + 14x_2 \le 155$$
$$x_1 \ge 0, \ x_2 \ge 0$$

Then, the efficient solution of $QP_\alpha^{uu}(\omega^*)$ is 2888

$x_1 = 19$, $x_2 = 0$

And the maximum value of $QP_\alpha^{uu}(x^*) = 2888$

$$\max \ QP_\alpha^{u\ell}(\omega^*) = (7.125x_1^2 + 13.75x_1x_2 + 7x_2^2)$$
$$s.t.$$
$$6x_1 + 9x_2 \le 90$$
$$9x_1 + 15x_2 \le 130$$
$$x_1 \ge 0, x_2 \ge 0$$

Then, the efficient solution of $QP_\alpha^{u\ell}(\omega^*)$ is 1486.574

$x_1 = 14.444$, $x_2 = 0$

And the maximum value of $QP_\alpha^{u\ell}(x^*) = 1486.574$

$$\max \ QP_\alpha^{\ell\ell}(\omega^*) = (4.875x_1^2 + 8x_1x_2 + 5.5x_2^2)$$
$$s.t.$$
$$12x_1 + 15x_2 \le 65$$
$$15x_1 + 28x_2 \le 115$$
$$x_1 \ge 0, \ x_2 \ge 0$$

Then, the efficient solution of $QP_\alpha^{\ell\ell}(\omega^*)$ is 143.03385

$x_1 = 5.4167$, $x_2 = 0$

And the maximum value of $QP_\alpha^{\ell\ell}(x^*) = 143.03385$

$$\max \ QP_\alpha^{\ell u}(\omega^*) = (4.125x_1^2 + 7.5x_1x_2 + 5x_2^2)$$
$$s.t.$$
$$14x_1 + 20x_2 \le 62.5$$
$$20x_1 + 40x_2 \le 75$$
$$x_1 \ge 0, \ x_2 \ge 0$$

Then, the efficient solution of $QP_\alpha^{\ell u}(\omega^*)$ is 58.0078

$x_1 = 3.75$, $x_2 = 0$

And the maximum value of $QP_\alpha^{\ell u}(x^*) = 58.0078$

4. α-Cut Approach for FFRMOQP Problem

The multi-objective quadratic programming problems with fully fuzzy rough coefficients and decision variables are defined as follows:

$$\left.\begin{array}{l} \max MOQP_r^R(x^R) = \sum_{j=1}^{n} \tilde{c}_{r_j}^R x_j^R + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_{r_{ij}}^R x_i^R x_j^R, r = 1, 2, ..., k \\ \text{Subject to} \\ \sum_{j=1}^{n} a_{ij}^R x_j^R \le \tilde{b}_i^R, \\ x_j^R \ge 0, i \in I = \{1, 2, ..., n\}; \ j \in J = \{1, 2, ..., n\} \end{array}\right\} \quad (13)$$

Where $\tilde{c}_{r_j}^R$, $\tilde{a}_{ij}^R$, $\tilde{b}_i^R$, $\tilde{q}_{r_{ij}}^R$ and $\tilde{x}_j^R$

Are fuzzy rough coefficient and variables, respectively, $i\in I$, $j\in J$, $r=1,\dots,k$.

Problem (13) can be converted as:

$$\left.\begin{array}{l}(FFRMOQP): max[\,\widetilde{QP}_r^L, \widetilde{QP}_r^U]\\ = \sum_{j=1}^{n}[\tilde{c}_{r_j}^L : \tilde{c}_{r_j}^U] \otimes [\tilde{x}_j^L : \tilde{x}_j^U] + \\ \qquad \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[\tilde{q}_{r_{ij}}^L : \tilde{q}_{r_{ij}}^U] \otimes [\tilde{x}_i^L : \tilde{x}_i^U] \otimes [\tilde{x}_j^L : \tilde{x}_j^U] \\ \qquad\qquad ,r = 1,2,\dots,k \\ \text{S.t} \\ \quad \sum_{j=1}^{n}[\tilde{a}_{ij}^L : \tilde{a}_{ij}^U] \otimes [\tilde{x}_j^L : \tilde{x}_j^U] \le [\tilde{b}_i^L : \tilde{b}_i^U], \\ \quad \tilde{x}_j^L, \tilde{x}_j^U \ge 0\,, i \in I = \{1,2,\dots,n\}; \\ \quad j \in J = \{1,2,\dots,n\}\end{array}\right\} \quad (14)$$

Such that $\tilde{c}_{r_j}^{LAI}, \tilde{c}_{r_j}^{UAI}, \tilde{q}_{r_{ij}}^{LAI}, \tilde{q}_{r_{ij}}^{UAI}, \tilde{a}_{ij}^{LAI}, \tilde{a}_{ij}^{UAI}, \tilde{b}_i^{LAI}, \tilde{b}_i^{UAI}$

Are triangular fuzzy numbers , x_j^L *and* x_j^U are triangular fuzzy variables

Where

$\tilde{c}_{rj}^L = (c_{rj}^{\ell\ell}, c_{rj}^m, c_{rj}^{u\ell})$, $\quad \tilde{c}_{rj}^U = (c_{rj}^{\ell u}, c_{rj}^m, c_{rj}^{uu})$,
$\tilde{a}_{ij}^L = (a_{ij}^{\ell\ell}, a_{ij}^m, a_{ij}^{u\ell})$, $\quad \tilde{a}_{ij}^U = (a_{ij}^{\ell u}, a_{ij}^m, a_{ij}^{uu})$,
$\tilde{b}_i^L = (b_i^{\ell\ell}, b_i^m, b_i^{u\ell})$, $\quad \tilde{b}_i^U = (b_i^{\ell u}, b_i^m, b_i^{uu})$,
$\tilde{q}_{rij}^L = (q_{rij}^{\ell\ell}, q_{rij}^m, q_{rij}^{u\ell})$, $\quad \tilde{q}_{rij}^U = (q_{rij}^{\ell u}, q_{rij}^m, q_{rij}^{uu})$.
$\tilde{x}_j^L = (x_j^{\ell\ell}, x_j^m, x_j^{u\ell})$, $\quad \tilde{x}_j^U = (x_j^{\ell u}, x_j^m, x_j^{uu})$,
$\tilde{x}_i^L = (x_i^{\ell\ell}, x_i^m, x_i^{u\ell})$, $\quad \tilde{x}_i^U = (x_i^{\ell u}, x_i^m, x_i^{uu})$.

According definition 2.13.we used α-cut (α-level set) of coefficient as:

$$(\tilde{c}_{rj}^L)_\alpha = ((c_{rj}^{\ell\ell})_\alpha, (c_{rj}^{u\ell})_\alpha), \quad (\tilde{c}_{rj}^U)_\alpha = ((c_{rj}^{\ell u})_\alpha, (c_{rj}^{uu})_\alpha),$$
$$(\tilde{c}_{rj}^L)_\alpha = [c_{rj}^{\ell\ell} + (c_{rj}^m - c_{rj}^{\ell\ell})\,\alpha, c_{rj}^{u\ell} + (c_{rj}^m - c_{rj}^{u\ell})\alpha]$$
$$(\tilde{c}_{rj}^U)_\alpha = [c_{rj}^{\ell u} + (c_{rj}^m - c_{rj}^{\ell u})\,\alpha, c_{rj}^{uu} + (c_{rj}^m - c_{rj}^{uu})\alpha]$$
$$(\tilde{a}_{ij}^L)_\alpha = ((a_{ij}^{\ell\ell})_\alpha, (a_{ij}^{u\ell})_\alpha), \quad (\tilde{a}_{ij}^U)_\alpha = ((a_{ij}^{\ell u})_\alpha, (a_{ij}^{uu})_\alpha),$$
$$(\tilde{a}_{ij}^L)_\alpha = [a_{ij}^{\ell\ell} + (a_{ij}^m - a_{ij}^{\ell\ell})\,\alpha, a_{ij}^{u\ell} + (a_{ij}^m - a_{ij}^{u\ell})\,\alpha]$$
$$(\tilde{a}_{ij}^U)_\alpha = [a_{ij}^{\ell u} + (a_{ij}^m - a_{ij}^{\ell u})\,\alpha, a_{ij}^{uu} + (a_{ij}^m - a_{ij}^{uu})\,\alpha]$$
$$(\tilde{b}_i^L)_\alpha = ((b_i^{\ell\ell})_\alpha, (b_i^{u\ell})_\alpha), \quad (\tilde{b}_i^U)_\alpha = ((b_i^{\ell u})_\alpha, (b_i^{uu})_\alpha),$$
$$(\tilde{b}_i^L)_\alpha = [b_i^{\ell\ell} + (b_i^m - b_i^{\ell\ell})\,\alpha, b_i^{u\ell} + (b_i^m - b_i^{u\ell})\,\alpha]$$
$$(\tilde{b}_i^U)_\alpha = [b_i^{\ell u} + (b_i^m - b_i^{\ell u})\,\alpha, b_i^{uu} + (b_i^m - b_i^{uu})\,\alpha]$$
$$(\tilde{q}_{rij}^L)_\alpha = ((q_{rij}^{\ell\ell})_\alpha, (q_{rij}^{u\ell})_\alpha), \quad (\tilde{q}_{rij}^U)_\alpha = ((q_{rij}^{\ell u})_\alpha, (q_{rij}^{uu})_\alpha).$$
$$(\tilde{q}_{rij}^L)_\alpha = [q_{rij}^{\ell\ell} + (q_{rij}^m - q_{rij}^{\ell\ell})\,\alpha, q_{rij}^{u\ell} + (q_{rij}^m - q_{rij}^{u\ell})\,\alpha]$$
$$(\tilde{q}_{rij}^U)_\alpha = [q_{rij}^{\ell u} + (q_{rij}^m - q_{rij}^{\ell u})\,\alpha, q_{rij}^{uu} + (q_{rij}^m - q_{rij}^{uu})\,\alpha]$$

To convert FFRMOQPP to FRMOQPP and get as:

$$\left.\begin{array}{l}\mathrm{FRMOQP}_{r\alpha}^R = \\ \max(\sum_{j=1}^{n}[[(c_{r_j}^{\ell\ell})_\alpha,(c_{r_j}^{u\ell})_\alpha]:[(c_{r_j}^{\ell u})_\alpha,(c_{r_j}^{uu})_\alpha]]\otimes[((x_j^{\ell\ell})_\alpha,(x_j^{u\ell})_\alpha):((x_j^{\ell u})_\alpha,(x_j^{uu})_\alpha)] \\ \quad +\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[[(q_{r_{ij}}^{\ell\ell})_\alpha,(q_{r_{ij}}^{u\ell})_\alpha]:[(q_{r_{ij}}^{\ell u})_\alpha,(q_{r_{ij}}^{uu})_\alpha]]\otimes[[(x_i^{\ell\ell})_\alpha,(x_i^{u\ell})_\alpha]:[(x_i^{\ell u})_\alpha,(x_i^{uu})_\alpha]]\otimes \\ \qquad [[(x_j^{\ell\ell})_\alpha,(x_j^{u\ell})_\alpha]:[(x_j^{\ell u})_\alpha,(x_j^{uu})_\alpha]],\ r=1,2,\dots,k \\ \quad \text{Subject to} \\ \quad \sum_{j=1}^{n}[[(a_{ij}^{\ell\ell})_\alpha,(a_{ij}^{u\ell})_\alpha]:[(a_{ij}^{\ell u})_\alpha,(a_{ij}^{uu})_\alpha]]\otimes[[(x_j^{\ell\ell})_\alpha,(x_j^{u\ell})_\alpha]:[(x_j^{\ell u})_\alpha,(x_j^{uu})_\alpha]], \\ \qquad \le[[(b_i^{\ell\ell})_\alpha,\ (b_i^{u\ell})_\alpha]:[(b_i^{\ell u})_\alpha,(b_i^{uu})_\alpha]] \\ \qquad i\in I=\{1,2,\dots,n\}; \\ \qquad j\in J=\{1,2,\dots,n\},\alpha\in(0,1]\end{array}\right\}(15)$$

Therefore, we can write (15) as:

$$\left.\begin{array}{l}\max\ \mathrm{FRMOQP}_{r\alpha}^R = \\ \sum_{j=1}^{n}[[(c_{r_j}^{\ell\ell})_\alpha,(c_{r_j}^{u\ell})_\alpha]:[(c_{r_j}^{\ell u})_\alpha,(c_{r_j}^{uu})_\alpha]]\otimes[((x_j^{\ell\ell})_\alpha,(x_j^{u\ell})_\alpha):((x_j^{\ell u})_\alpha,(x_j^{uu})_\alpha)] \\ \quad +\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[[(q_{r_{ij}}^{\ell\ell})_\alpha,(q_{r_{ij}}^{u\ell})_\alpha]:[(q_{r_{ij}}^{\ell u})_\alpha,(q_{r_{ij}}^{uu})_\alpha]]\otimes[[(x_i^{\ell\ell})_\alpha,(x_i^{u\ell})_\alpha]:[(x_i^{\ell u})_\alpha,(x_i^{uu})_\alpha]] \\ \qquad \otimes[[(x_j^{\ell\ell})_\alpha,(x_j^{u\ell})_\alpha]:[(x_i^{\ell u})_\alpha,(x_j^{uu})_\alpha]]\ ,r=1,2,\dots,k \\ \quad \text{Subject to} \\ \quad \sum_{j=1}^{n}[[(a_{ij}^{\ell\ell})_\alpha,(a_{ij}^{u\ell})_\alpha]:[(a_{ij}^{\ell u})_\alpha,(a_{ij}^{uu})_\alpha]]\otimes[[(x_j^{\ell\ell})_\alpha,(x_j^{u\ell})_\alpha]:[(x_j^{\ell u})_\alpha,(x_j^{uu})_\alpha]] \\ \qquad \le[[(b_i^{\ell\ell})_\alpha,(b_i^{u\ell})_\alpha]:[(b_i^{\ell u})_\alpha,(b_i^{uu})_\alpha]] \\ \quad i\in I=\{1,2,\dots,n\};\ \ j\in J=\{1,2,\dots,n\},\alpha\in(0,1]\end{array}\right\}(16)$$

Where

$$FRMOQP_{r\alpha}^R(x)=[\mathrm{QP}_{r\alpha}^{\ell\ell}(x),\mathrm{QP}_{r\alpha}^{u\ell}(\mathrm{x}):\mathrm{QP}_{r\alpha}^{\ell u}(x),\mathrm{QP}_{r\alpha}^{uu}(\mathrm{x})].$$

Therefore the problem (16) decomposes to a multi-objective quadratic programming problem defines as follows:

$$\left.\begin{array}{l}\max\mathrm{MOQP}_{r\alpha}^{uu}(x)=\sum_{j=1}^{n}(c_{r_j}^{uu})_\alpha\otimes(x_j^{uu})_\alpha\ +\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{r_{ij}}^{uu})_\alpha\otimes(x_i^{uu})_\alpha\otimes(x_j^{uu})_\alpha \\ \qquad\qquad r=1,2,\dots,k \\ \quad \text{Subject to} \\ \quad \sum_{j=1}^{n}(a_{ij}^{uu})_\alpha\otimes(x_j^{uu})_\alpha\le(b_i^{uu})_\alpha \\ \quad (x_j^{uu})_\alpha\ge 0,\ i\in I=\{1,2,\dots,n\}; \\ \quad j\in J=\{1,2,\dots,n\},\ \alpha\in(0,1]\end{array}\right\}(17)$$

$$\left.\begin{array}{l}\max\mathrm{MOQP}_{r\alpha}^{u\ell}(x)= \\ \quad \sum_{j=1}^{n}(c_{r_j}^{u\ell})_\alpha\otimes(x_j^{u\ell})_\alpha+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{r_{ij}}^{u\ell})_\alpha\otimes(x_i^{u\ell})_\alpha\otimes(x_j^{u\ell})_\alpha \\ \qquad\qquad ,\ r=1,2,\dots,k \\ \quad \text{S.t} \\ \quad \sum_{j=1}^{n}(a_{ij}^{u\ell})_\alpha\otimes(x_j^{u\ell})_\alpha\le(b_i^{u\ell})_\alpha \\ \quad (x_j^{uu})_\alpha\ge(x_j^{u\ell})_\alpha;\ \ (x_j^{u\ell})_\alpha\ge 0 \\ \quad i\in I=\{1,2,\dots,n\};\ \ j\in J=\{1,2,\dots,n\},\ \alpha\in(0,1]\end{array}\right\}(18)$$

$$\left.\begin{array}{l} \max MOQP^{\ell\ell}_{r\alpha}(x) = \\ \sum_{j=1}^{n}(c^{\ell\ell}_{r_j})_\alpha \otimes (x^{\ell\ell}_j)_\alpha + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q^{\ell\ell}_{r_{ij}})_\alpha \otimes (x^{\ell\ell}_i)_\alpha \otimes (x^{\ell\ell}_j)_\alpha \\ \qquad ,r=1,2,\ldots,k \\ \text{S.t} \\ \sum_{j=1}^{n}(a^{\ell\ell}_{ij})_\alpha \otimes (x^{\ell\ell}_j)_\alpha \le (b^{\ell\ell}_i)_\alpha \\ (x^{u\ell}_j)_\alpha \ge (x^{\ell\ell}_j)_\alpha;\ (x^{\ell\ell}_j)_\alpha \ge 0 \\ i \in I=\{1,2,\ldots,n\};\ j \in J=\{1,2,\ldots,n\}, \alpha \in (0,1] \end{array}\right\} \quad (19)$$

$$\left.\begin{array}{l} \max MOQP^{\ell u}_{r\alpha}(x) = \\ \sum_{j=1}^{n}(c^{\ell u}_{r_j})_\alpha \otimes (x^{\ell u}_j)_\alpha + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q^{\ell u}_{r_{ij}})_\alpha \otimes (x^{\ell u}_i)_\alpha \otimes (x^{\ell u}_j)_\alpha \\ \qquad ,r=1,2,\ldots,k \\ \text{S.t} \\ \sum_{j=1}^{n}(a^{\ell u}_{ij})_\alpha \otimes (x^{\ell u}_j)_\alpha \le (b^{\ell u}_i)_\alpha \\ (x^{\ell\ell}_j)_\alpha \ge (x^{\ell u}_j)_\alpha;\ (x^{\ell u}_j)_\alpha \ge 0 \\ i \in I=\{1,2,\ldots,n\};\ j \in J=\{1,2,\ldots,n\}, \alpha \in (0,1] \end{array}\right\} \quad (20)$$

Now, using the weighted sum method to convert the MOQP problem to quadratic programming QP problem:

$$\left.\begin{array}{l} QP^{uu}_\alpha(\omega^*): \max QP^{uu}_{r\alpha}(x) = \\ \sum_{r=1}^{k}\omega_r[\sum_{j=1}^{n}(c^{uu}_{r_j})_\alpha \otimes (x^{uu}_j)_\alpha + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q^{uu}_{r_{ij}})_\alpha \otimes (x^{uu}_i)_\alpha \otimes (x^{uu}_j)_\alpha] \\ \qquad ,r=1,2,\ldots,k \\ \text{Subject to} \\ \sum_{j=1}^{n}(a^{uu}_{ij})_\alpha \otimes (x^{uu}_j)_\alpha \le (b^{uu}_i)_\alpha \\ (x^{uu}_j)_\alpha \ge 0 \\ i \in I=\{1,2,\ldots,n\}; \\ j \in J=\{1,2,\ldots,n\}, \alpha \in (0,1], \omega_r \in w \end{array}\right\} \quad (21)$$

$$\left.\begin{array}{l} QP^{u\ell}_\alpha(\omega^*): \max QP^{u\ell}_{r\alpha}(x) = \\ \sum_{r=1}^{k}\omega_r[\sum_{j=1}^{n}(c^{u\ell}_{r_j})_\alpha \otimes (x^{u\ell}_j)_\alpha + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q^{u\ell}_{r_{ij}})_\alpha \otimes (x^{u\ell}_i)_\alpha \otimes (x^{u\ell}_j)_\alpha] \\ \qquad ,r=1,2,\ldots,k \\ \text{Subject to} \\ \sum_{j=1}^{n}(a^{u\ell}_{ij})_\alpha \otimes (x^{u\ell}_j)_\alpha \le (b^{u\ell}_i)_\alpha \\ (x^{uu}_j)_\alpha \ge (x^{u\ell}_j)_\alpha;\ (x^{u\ell}_j)_\alpha \ge 0 \\ i \in I=\{1,2,\ldots,n\}; \\ j \in J=\{1,2,\ldots,n\},\ \alpha \in (0,1], \omega_r \in w \end{array}\right\} \quad (22)$$

$$\left.\begin{array}{l} QP^{\ell\ell}_\alpha(\omega^*): \max QP^{\ell\ell}_{r\alpha}(x) = \\ \sum_{r=1}^{k}\omega_r[\sum_{j=1}^{n}(c^{\ell\ell}_{r_j})_\alpha \otimes (x^{\ell\ell}_j)_\alpha + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q^{\ell\ell}_{r_{ij}})_\alpha \otimes (x^{\ell\ell}_i)_\alpha \otimes (x^{\ell\ell}_j)_\alpha] \\ \qquad ,r=1,2,\ldots,k \\ \text{Subject to} \\ \sum_{j=1}^{n}(a^{\ell\ell}_{ij})_\alpha \otimes (x^{\ell\ell}_j)_\alpha \le (b^{\ell\ell}_i)_\alpha \\ (x^{u\ell}_j)_\alpha \ge (x^{\ell\ell}_j)_\alpha;\ (x^{\ell\ell}_j)_\alpha \ge 0 \\ i \in I=\{1,2,\ldots,n\}; \\ j \in J=\{1,2,\ldots,n\}, \alpha \in (0,1], \omega_r \in w \end{array}\right\} \quad (23)$$

$$\left.\begin{array}{l} QP^{\ell u}_\alpha(\omega^*): \max QP^{\ell u}_{r\alpha}(x) = \\ \sum_{r=1}^{k}\omega_r[\sum_{j=1}^{n}(c^{\ell u}_{r_j})_\alpha \otimes (x^{\ell u}_j)_\alpha + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q^{\ell u}_{r_{ij}})_\alpha \otimes (x^{\ell u}_i)_\alpha \otimes (x^{\ell u}_j)_\alpha] \\ \qquad ,r=1,2,\ldots,k \\ \text{Subject to} \\ \sum_{j=1}^{n}(a^{\ell u}_{ij})_\alpha \otimes (x^{\ell u}_j)_\alpha \le (b^{\ell u}_i)_\alpha \\ (x^{\ell\ell}_j)_\alpha \ge (x^{\ell u}_j)_\alpha;\ (x^{\ell u}_j)_\alpha \ge 0 \\ i \in I=\{1,2,\ldots,n\}; \\ j \in J=\{1,2,\ldots,n\}, \alpha \in (0,1], \omega_r \in w \end{array}\right\} \quad (24)$$

The optimal solutions of

$QP^{uu}_\alpha(\omega^*), QP^{u\ell}_\alpha(\omega^*), QP^{\ell\ell}_\alpha(\omega^*)$ and $QP^{\ell u}_\alpha(\omega^*)$ Problems are the efficient solutions of

$MOQP^{uu}_\alpha, MOQP^{u\ell}_\alpha, MOQP^{\ell\ell}_\alpha$ *and* $MOQP^{\ell u}_\alpha$ Problems, respectively.

α-Cut Algorithm for FFRMOQP problem

1) Convert the problem to the form (13) .
2) Use α-level cuts to deal with a fuzziness of fully fuzzy rough coefficients and decision variables as a form (15).
3) Use decompose technique for (14)problem (15)to get the MOQP problem

   $MOQP^{uu}_\alpha, MOQP^{u\ell}_\alpha, MOQP^{\ell\ell}_\alpha$ *and* $MOQP^{\ell u}_\alpha$
4) Use one of the secularization methods; say the weights method, to convert each problem $MOQP^{uu}_\alpha, MOQP^{u\ell}_\alpha, MOQP^{\ell\ell}_\alpha$ *and* $MOQP^{\ell u}_\alpha$ with a single objective in the form $QP^{uu}_\alpha(\omega^*), QP^{u\ell}_\alpha(\omega^*), QP^{\ell\ell}_\alpha(\omega^*)$ and $QP^{\ell u}_\alpha(\omega^*)$

   For $\omega = \omega^* \in W$ Find the optimal solution of each quadratic programming problem $QP^{uu}_\alpha(\omega^*), QP^{u\ell}_\alpha(\omega^*), QP^{\ell\ell}_\alpha(\omega^*)$ and $QP^{\ell u}_\alpha(\omega^*)$.

Example 2: Consider the following FFRMOQP

$$\text{Max } QP_r^R(x)=$$

$$\begin{pmatrix} \frac{1}{2}\begin{pmatrix}x_1^R & x_2^R\end{pmatrix}\otimes\begin{pmatrix}[[1,2,4]:[0.5,2,5]] & [[5,12,15]:[4,12,18]]\\ [[5,12,15]:[4,12,18]] & [[14,18,20]:[10,18,32]]\end{pmatrix}\otimes\begin{pmatrix}x_1^R\\ x_2^R\end{pmatrix}, \\ \frac{1}{2}\begin{pmatrix}x_1^R & x_2^R\end{pmatrix}\otimes\begin{pmatrix}[[3,6,14]:[1,6,22]] & [[4,18,20]:[2,18,26]]\\ [[4,18,20]:[2,18,26]] & [[10,20,40]:[8,20,60]]\end{pmatrix}\otimes\begin{pmatrix}x_1^R\\ x_2^R\end{pmatrix}\end{pmatrix}$$

$s.t$

$$\begin{pmatrix}[[3,6,9]:[2,6,10]] & [[4,8,10]:[2,8,16]]\\ [[6,16,22]:[4,16,28]] & [[9,18,22]:[8,18,24]]\end{pmatrix}\otimes\begin{pmatrix}x_1^R\\ x_2^R\end{pmatrix} \le \begin{pmatrix}[[90,100,130]:[50,100,180]]\\ [[70,140,200]:[60,140,220]]\end{pmatrix}$$

$$\max\ QP_r^R=$$

$$\begin{pmatrix}
[[0.5,1,2]:[0.25,1,2.5]]\otimes[[(x_1^{\ell\ell})^2,(x_1^{m})^2,(x_1^{u\ell})^2]:[(x_1^{\ell u})^2,(x_1^{m})^2,(x_1^{uu})^2]] \\
+[[5,12,15]:[4,12,18]]\otimes[[(x_1^{\ell\ell}),(x_1^{m}),(x_1^{u\ell})]:[(x_1^{\ell u}),(x_1^{m}),(x_1^{uu})]] \\
\otimes[[(x_2^{\ell\ell}),(x_2^{m}),(x_2^{u\ell})]:[(x_2^{\ell u}),(x_2^{m}),(x_2^{uu})]] \\
+[[7,9,10]:[5,9,16]]\otimes[[(x_2^{\ell\ell})^2,(x_2^{m})^2,(x_2^{u\ell})^2]:[(x_2^{\ell u})^2,(x_2^{m})^2,(x_2^{uu})^2]]\ , \\
[[1.5,3,7]:[0.5,3,11]]\otimes[[(x_1^{\ell\ell})^2,(x_1^{m})^2,(x_1^{u\ell})^2]:[(x_1^{\ell u})^2,(x_1^{m})^2,(x_1^{uu})^2]] \\
+[[4,18,20]:[2,18,26]]\otimes[[(x_1^{\ell\ell}),(x_1^{m}),(x_1^{u\ell})]:[(x_1^{\ell u}),(x_1^{m}),(x_1^{uu})]] \\
\otimes[[(x_2^{\ell\ell}),(x_2^{m}),(x_2^{u\ell})]:[(x_2^{\ell u}),(x_2^{m}),(x_2^{uu})]] \\
+[[5,10,20]:[4,10,30]]\otimes[[(x_2^{\ell\ell})^2,(x_2^{m})^2,(x_2^{u\ell})^2]:[(x_2^{\ell u})^2,(x_2^{m})^2,(x_2^{uu})^2]]
\end{pmatrix}$$

$s.t.$

$$[[3,6,9]:[2,6,10]]\otimes[[(x_1^{\ell\ell}),(x_1^{m}),(x_1^{u\ell})]:[(x_1^{\ell u}),(x_1^{m}),(x_1^{uu})]]+$$
$$[[4,8,10]:[2,8,16]]\otimes[[(x_2^{\ell\ell}),(x_2^{m}),(x_2^{u\ell})]:[(x_2^{\ell u}),(x_2^{m}),(x_2^{uu})]]$$
$$\le[[90,100,130]:[50,100,180]]$$
$$[[6,16,22]:[4,16,28]]\otimes[[(x_1^{\ell\ell}),(x_1^{m}),(x_1^{u\ell})]:[(x_1^{\ell u}),(x_1^{m}),(x_1^{uu})]]+$$
$$[[9,18,22]:[8,18,24]]\otimes[[(x_2^{\ell\ell}),(x_2^{m}),(x_2^{u\ell})]:[(x_2^{\ell u}),(x_2^{m}),(x_2^{uu})]]$$
$$\le[[70,140,200]:[60,140,220]]$$
$$x_1^{\ell\ell},x_1^{m},x_1^{u\ell},x_1^{\ell u},x_1^{uu},x_2^{\ell\ell},x_2^{m},x_2^{u\ell},x_2^{\ell u},x_2^{uu}\ge 0$$

For $\alpha=0.5$ we get:

$$\max\ QP_r^R=$$

$$\begin{pmatrix}
[[.75,1.5]:[.625,1.75]]\otimes[[(x_1^{\ell\ell})_\alpha^2,(x_1^{u\ell})_\alpha^2]:[(x_1^{\ell u})_\alpha^2,(x_1^{uu})_\alpha^2]] \\
+[[8.5,13.5]:[8,15]]\otimes[[(x_1^{\ell\ell})_\alpha,(x_1^{u\ell})_\alpha]:[(x_1^{\ell u})_\alpha,(x_1^{uu})_\alpha]] \\
\otimes[[(x_2^{\ell\ell})_\alpha,(x_2^{u\ell})_\alpha]:[(x_2^{\ell u})_\alpha,(x_2^{uu})_\alpha]] \\
+[[8,9.5]:[7,12.5]]\otimes[[(x_2^{\ell\ell})_\alpha^2,(x_2^{u\ell})_\alpha^2]:[(x_2^{\ell u})_\alpha^2,(x_2^{uu})_\alpha^2]]\ , \\
[[2.25,5]:[1.75,7]]\otimes[[(x_1^{\ell\ell})_\alpha^2,(x_1^{u\ell})_\alpha^2]:[(x_1^{\ell u})_\alpha^2,(x_1^{uu})_\alpha^2]] \\
+[[11,19]:[10,22]]\otimes[[(x_1^{\ell\ell})_\alpha,(x_1^{u\ell})_\alpha]:[(x_1^{\ell u})_\alpha,(x_1^{uu})_\alpha]] \\
\otimes[[(x_2^{\ell\ell})_\alpha,(x_2^{u\ell})_\alpha]:[(x_2^{\ell u})_\alpha,(x_2^{uu})_\alpha]] \\
+[[7.5,15]:[7,20]]\otimes[[(x_2^{\ell\ell})_\alpha^2,(x_2^{u\ell})_\alpha^2]:[(x_2^{\ell u})_\alpha^2,(x_2^{uu})_\alpha^2]]
\end{pmatrix}$$

$s.t.$

$$[[4.5,7.5]:[4,8]]\otimes[[(x_1^{\ell\ell})_\alpha,(x_1^{u\ell})_\alpha]:[(x_1^{\ell u})_\alpha,(x_1^{uu})_\alpha]]$$
$$+[[6,9]:[5,12]]\otimes[[(x_2^{\ell\ell})_\alpha,(x_2^{u\ell})_\alpha]:[(x_2^{\ell u})_\alpha,(x_2^{uu})_\alpha]]$$
$$\le[[95,115]:[75,140]]$$
$$[[11,19]:[10,22]]\otimes[[(x_1^{\ell\ell})_\alpha,(x_1^{u\ell})_\alpha]:[(x_1^{\ell u})_\alpha,(x_1^{uu})_\alpha]]$$
$$+[[13.5,20]:[13,21]]\otimes[[(x_2^{\ell\ell})_\alpha,(x_2^{u\ell})_\alpha]:[(x_2^{\ell u})_\alpha,(x_2^{uu})_\alpha]]$$
$$\le[[105,170]:[100,180]]$$

The above problem can be written as the following problems:

$$MOQP_\alpha^{uu}: max\ \ \mathrm{MOQP}_\alpha^{uu}=$$
$$\begin{pmatrix}1.75(x_1^{uu})_\alpha^2+15(x_1^{uu})_\alpha(x_2^{uu})_\alpha+12.5(x_2^{uu})_\alpha^2\ ,\\ 7(x_1^{uu})_\alpha^2+22(x_1^{uu})_\alpha(x_2^{uu})_\alpha+20(x_2^{uu})_\alpha^2\end{pmatrix}$$
$s.t.$
$$8(x_1^{uu})_\alpha+12(x_2^{uu})_\alpha\le 140$$
$$22(x_1^{uu})_\alpha+21(x_2^{uu})_\alpha\le 180$$
$$(x_1^{uu})_\alpha\ \ge 0\ ,(x_2^{uu})_\alpha\ \ge 0$$

$$MOQP_\alpha^{u\ell}: max\ \ \mathrm{MOQP}_\alpha^{u\ell}=$$
$$\begin{pmatrix}1.5(x_1^{u\ell})_\alpha^2+13.5(x_1^{u\ell})_\alpha(x_2^{u\ell})_\alpha+9.5(x_2^{u\ell})_\alpha^2\ ,\\ 5(x_1^{u\ell})_\alpha^2+19(x_1^{u\ell})_\alpha(x_2^{u\ell})_\alpha+15(x_2^{u\ell})_\alpha^2\end{pmatrix}$$
$s.t.$
$$7.5(x_1^{u\ell})_\alpha+9(x_2^{u\ell})_\alpha\le 115$$
$$19(x_1^{u\ell})_\alpha+20(x_2^{u\ell})_\alpha\le 170$$
$$(x_1^{u\ell})_\alpha\ge 0\ ,(x_1^{u\ell})_\alpha\ \ge 0$$

$$MOQP_\alpha^{\ell\ell}: max\ \ \mathrm{MOQP}_\alpha^{\ell\ell}=$$
$$\begin{pmatrix}0.75(x_1^{\ell\ell})_\alpha^2+8.5(x_1^{\ell\ell})_\alpha(x_2^{\ell\ell})_\alpha+8(x_2^{\ell\ell})_\alpha^2\ ,\\ 2.25(x_1^{\ell\ell})_\alpha^2+11(x_1^{\ell\ell})_\alpha(x_2^{\ell\ell})_\alpha+7.5(x_2^{\ell\ell})_\alpha^2\end{pmatrix}$$
$s.t.$
$$4.5(x_1^{\ell\ell})_\alpha+6(x_2^{\ell\ell})_\alpha\le 95$$
$$11(x_1^{\ell\ell})_\alpha+13.5(x_2^{\ell\ell})_\alpha\le 105$$
$$(x_1^{\ell\ell})_\alpha\ge 0,(x_2^{\ell\ell})_\alpha\ge 0$$

$$MOQP_\alpha^{\ell u}: max\ \ \mathrm{MOQP}_\alpha^{\ell u}=$$
$$\begin{pmatrix}0.625(x_1^{\ell u})_\alpha^2+8(x_1^{\ell u})_\alpha(x_2^{\ell u})_\alpha+7(x_2^{\ell u})_\alpha^2\ ,\\ 1.75(x_1^{\ell u})_\alpha^2+10(x_1^{\ell u})_\alpha(x_2^{\ell u})_\alpha+7(x_2^{\ell u})_\alpha^2\end{pmatrix}$$
$s.t.$
$$4(x_1^{\ell u})_\alpha+5(x_2^{\ell u})_\alpha\le 75$$
$$10(x_1^{\ell u})_\alpha+13(x_2^{\ell u})_\alpha\le 100$$
$$(x_1^{\ell u})_\alpha\ge 0,(x_2^{\ell u})_\alpha\ge 0$$

For $\omega_1=\omega_2=.5$ we getting the corresponded weighting quadratic programming problems

$QP_\alpha^{uu}(\omega^*), QP_\alpha^{u\ell}(\omega^*), QP_\alpha^{\ell\ell}(\omega^*)$ and $\mathrm{QP}_\alpha^{\ell u}(\omega^*)$For $MOQP_\alpha^{uu}, MOQP_\alpha^{u\ell}, MOQP_\alpha^{\ell\ell}$ and $\mathrm{MOQP}_\alpha^{\ell u}$, respectively, can be described as follows:

$$QP_\alpha^{uu}: max\ Q\ P_\alpha^{uu}=$$
$$4.375(x_1^{uu})_\alpha^2+18.5(x_1^{uu})_\alpha(x_2^{uu})_\alpha+16.25(x_2^{uu})_\alpha^2$$
$s.t.$
$$8(x_1^{uu})_\alpha+12(x_2^{uu})_\alpha\le 140$$
$$22(x_1^{uu})_\alpha+21(x_2^{uu})_\alpha\le 180$$
$$(x_1^{uu})_\alpha\ \ge 0\ ,(x_2^{uu})_\alpha\ \ge 0$$

Then, the efficient solution of $QP_\alpha^{uu}(\omega^*)$ is $\dfrac{58500}{49}$

$$(x_1^{uu})_\alpha = 0 \qquad , (x_2^{uu})_\alpha = \frac{60}{7}$$

And the maximum value of $QP_\alpha^{uu}(x^{*uu}) = \dfrac{58500}{49}$

$$QP_\alpha^{u\ell}: max \ \ QP_\alpha^{u\ell} = 3.25(x_1^{u\ell})_\alpha^2 + 16.25(x_1^{u\ell})_\alpha (x_2^{u\ell})_\alpha + 12.25(x_2^{u\ell})_\alpha^2$$
$$s.t.$$
$$7.5(x_1^{u\ell})_\alpha + 9(x_2^{u\ell})_\alpha \le 115$$
$$19(x_1^{u\ell})_\alpha + 20(x_2^{u\ell})_\alpha \le 170$$
$$(x_1^{u\ell})_\alpha \ge 0 , (x_1^{u\ell})_\alpha \ge 0$$

Then, the efficient solution of $MOQP_\alpha^{u\ell}$ is 885.063

$$(x_1^{*u\ell})_\alpha = 0 \qquad , (x_2^{*u\ell})_\alpha = 8.5$$

And the maximum value of $QP_\alpha^{u\ell}(x^{*u\ell}) = 885.063$

$$QP_\alpha^{\ell\ell}: max \ \ QP_\alpha^{\ell\ell} = 1.5(x_1^{\ell\ell})_\alpha^2 + 9.75(x_1^{\ell\ell})_\alpha (x_2^{\ell\ell})_\alpha + 7.75(x_2^{\ell\ell})_\alpha^2$$
$$s.t.$$
$$4.5(x_1^{\ell\ell})_\alpha + 6(x_2^{\ell\ell})_\alpha \le 95$$
$$11(x_1^{\ell\ell})_\alpha + 13.5(x_2^{\ell\ell})_\alpha \le 105$$
$$(x_1^{\ell\ell})_\alpha \ge 0, (x_2^{\ell\ell})_\alpha \ge 0$$

So, the efficient solution of $MOQP_\alpha^{\ell\ell}$ is 468.827

$$(x_1^{*\ell\ell})_\alpha = 0 \qquad , (x_2^{*\ell\ell})_\alpha = 7.78$$

And the maximum value of $QP_\alpha^{\ell\ell}(x^{*\ell\ell}) = 468.827$

$$QP_\alpha^{\ell u}: max \ \ QP_\alpha^{\ell u} = 1.1875(x_1^{\ell u})_\alpha^2 + 9(x_1^{\ell u})_\alpha (x_2^{\ell u})_\alpha + 7(x_2^{\ell u})_\alpha^2$$
$$s.t.$$
$$4(x_1^{\ell u})_\alpha + 5(x_2^{\ell u})_\alpha \le 75$$
$$10(x_1^{\ell u})_\alpha + 13(x_2^{\ell u})_\alpha \le 100$$
$$(x_1^{\ell u})_\alpha \ge 0, (x_2^{\ell u})_\alpha \ge 0$$

Then, the efficient solution of $MOQP_\alpha^{\ell u}$ is 414.2

$$(x_1^{*\ell u})_\alpha = 0 \qquad , (x_2^{*\ell u})_\alpha = \frac{100}{13}$$

And the maximum value of $QP_\alpha^{\ell u}(x^{*\ell u}) = 414.2$

Then, the $\alpha - cut$ fully fuzzy rough optimal solutions are:

$$(x_1^{*R})_\alpha = [(0,0):(0,0)],$$
$$(x_2^{*R})_\alpha = [(7.78,8.5):(7.69,8.57)].$$

Where the $\alpha - cut$ fuzzy rough efficient values range solutions for

$$\widetilde{QP}_\alpha^{*R} = [(468.827,885.063):(414.2,1193.88)].$$

And the α-cut fuzzy possibly optimal values range solution is

$$(QP_\alpha^{\ell u}(\omega^*), QP_\alpha^{uu}(\omega^*)) = (414.2,1193.88).$$

The α-cut fuzzy surely optimal values range solutions are:

$$(QP_\alpha^{\ell\ell}(\omega^*), QP_\alpha^{u\ell}(\omega^*)) = (468.827,885.063).$$

In addition, the α-cut completely satisfactory solutions are:

$$((x_1^{*\ell\ell})_\alpha, (x_1^{*u\ell})_\alpha) = (0,0),$$
$$((x_2^{*\ell\ell})_\alpha, (x_2^{*u\ell})_\alpha) = (7.78,8.5).$$

And the α-cut rather satisfactory solutions are:

$$((x_1^{*\ell u})_\alpha, (x_1^{*uu})_\alpha) = (0,0),$$
$$((x_2^{*\ell u})_\alpha, (x_2^{*uu})_\alpha) = (7.69,8.57).$$

4. Conclusion

In this paper, we introduced an α-Cut for solving two types of uncertainty multi-objective quadratic programming (MOQP) problems. The first one is fuzzy rough multi-objective quadratic programming (FRMOQP) problem. In this problem all coefficients in the objective and constraints functions are fuzzy rough. Second fully fuzzy rough multi-objective quadratic programming (FFRMOQP) problem .in this problem all coefficients and variables are fuzzy rough. We characterize the α-Cut fuzzy rough optimal solutions and the α-Cut fuzzy rough efficient values range solutions for (FRMOQP) problem by a given procedure. The algorithms of the two problems depend on Slice-sum method and the weighting sum method. The method has been validated by solving a number of practical problems. The solutions obtained aim the authors firstly to follow along this research line to try solving real problems in practice, in such a way that oriented Decision Support Systems involving Fuzzy rough multi-objective quadratic Programming problems can be built.

CRediT authorship contribution statement:

Conceptualization, Z.A. abotaleb and E. I. Ammar.; methodology, Z.A.abotaleb.; Software, Z.A. abotaleb; Validation, Z.A. abotaleb, E. I. Ammar. and A. Radwan.; formal analysis, E. I. Ammar.; investigation, A. Radwan.; resources, Z.A. abotaleb; data curation, Z.A. abotaleb.; writing—original draft preparation, Z.A. abotaleb.; writing—review and editing, Z.A. abotaleb.; visualization, Z.A. abotaleb.; supervision E. I. Ammar. ; Project administration, Z.A. abotaleb. Funding acquisition, Z.A. abotaleb. All authors have read and agreed to the published version of the manuscript.

Data availability statement

The data used to support the findings of this study are available from the corresponding author upon request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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