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Sumudu Transform Pade' Approximation Method for Solving Fractional Physical Models

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Abstract:This study offers a recent technique named the Sumudu Transform Pade' Approximation Method (STPAM) to treat fractional physical models. It comprises the Pade' Approximation Method (PAM) and the Sumudu Transform Method (STM).The Sumudu Transform Pade' Approximation Method (STPAM) enhances the accumulation rate of the truncated Maclaurin series by stratifying the Pade' method in the Sumudu transform method chain solution. The Caputo's fractional derivative was employed. It is necessary for simulating issues with non-local features and phenomena that account for interactions in the past. The Caputo fractional operator is more adaptable for analysis and can handle initial and boundary value issues. The principal objective of the study is to use the Sumudu Transform Pade' Approximation Method (STPAM) to solve fractional models that arise in physics. We solved fractional physical models using the Sumudu transform method (STM) and compared the results to the exact solutions and the approximate Pade' approximation method (PAM) to assess the quality of the Sumudu Transform Pade' Approximation Method (STPAM). The findings highlight STPAM's advantages, including its ease of use, effectiveness, universality, cleanliness, packability, quality, and clarity.
Keywords: Fractional calculus; Sumudu transform method; Pade´ approximation method; Mathematica software.

1. Introduction

Fractional differential equations (FDES) have received a lot of interest in the domains of physics, engineering, and other disciplines [1,2]. FDES must have analytical sequence solutions for several physical interactions in non-homogeneous mediums [3]. Numerous methods profile numerical and analytical solutions with fractional-order derivatives for physical phenomena stated as using differential equations, including the Sumudu transform (ST) approach [4], the domain decomposition Sumudu transform method (ADSTM) [5], the variation iteration Sumudu transform method (VISTM) [6], the homotopy Sumudu transform method (HSTM) [7,8], the adomain decomposition method (ADM) [9,10], the homotopy perturbation method (HPM) [11], the Laplace transform method (LT) [12-17], the homotopy analysis method (HAM) [18], the Q-homotopy analysis transform method (Q-HATM) [19], the natural transform method (NTM) [20–23], the Laplace residual power series method (LRPSM) [24], and the fractional exponential function (FEF) [25]. Here, we reminisce about various concepts.

Definition 1 ([26, 27]):
The Riemann Lowville (R-L) fractional integral factor of order ε, of a function $\kappa(\chi,t)$,is defined via :

$$J_t^{\varepsilon}\kappa(\chi,t)=\begin{cases}\dfrac{1}{\Gamma(\varepsilon)}\displaystyle\int_0^t (t-\tau)^{\varepsilon-1}\kappa(\chi,\tau)d\tau, \varepsilon>0, \chi\in I,\\ \kappa(\chi,t); \varepsilon=0, t<\tau\leq 0,\end{cases} \quad (1)$$

where Γ, is the gamma function and ε, is an arbitrary real number but fixed base point. For the R-L fractional integral we have: $I^{\varepsilon}t^{\lambda}=\frac{\Gamma(\lambda+1)}{\Gamma(\varepsilon+\lambda+1)}t^{\varepsilon+\lambda}$. (2)

Definition 2 ([26, 27]):
The Caputo fractional derivative (CFD) factor of order ε, of $\kappa(\chi,t)$, is defined via

$$D_t^{\varepsilon}\kappa(\chi,t)=\begin{cases}J_t^{\,m-\varepsilon}\left(\frac{\partial^m\kappa(\chi,t)}{\partial t^m}\right), m-1<\varepsilon\leq m, m\in \mathrm{N}\\ \frac{\partial^m\kappa(\chi,t)}{\partial t^m}, \varepsilon=m\end{cases}, \quad (3)$$

Definition 3 ([28, 29]):
The Mittag-Leffler Function (MLF) $E_{\alpha}(\eta)$ is defined via the series representation, valid in the whole complex plane as:

$$E_{\alpha}(\eta)=\sum_{k=0}^{\infty}\frac{\eta^k}{\Gamma(\alpha k+1)}, \alpha\in C, R(\alpha)>0, \eta\in \mathrm{C} \quad (4)$$

Definition 4 ([30]):
Watugala eta al. studied and offered The Sumudu transform (ST) as:

$$G(u)=S[\kappa(t);u]=\int_0^{\infty}\kappa(ut)e^{-t}\,dt, u\in(-t_1,t_2), \quad (5)$$

Definition 5 ([31]):
the Sumudu Transform (ST) of the Caputo fractional derivative is offered as:

$$S[D_t^{\varepsilon}v(t)]=u^{-\varepsilon}S[\kappa(t)]-\sum_{k=0}^{m-1}u^{-\varepsilon+k}\kappa^{(k)}(0+), (m-1<\varepsilon\leq m), \quad (6)$$

Definition 6 ([31]):
The duality Sumudu-Laplace transforms is:
Let $\kappa(t) \in A$, with Laplace Transform (LT) $K(s)$, then, the Sumudu Transform (ST) of $k(t)$ is given by:

$$G(u) = \frac{K(1/u)}{u}, \quad (7)$$

Definition 6 ([32- 34]): PAM symbolizes a function via:

$$\left[\frac{M}{\eta}\right] = \frac{a_0+a_1t^{\varepsilon}+a_2t^{2\varepsilon}+\ldots\ldots+a_Mt^{M\varepsilon}}{1+b_1t^{\varepsilon}+b_2\tau^{2\varepsilon}+\ldots\ldots+b_Z\tau^{\eta\varepsilon}}, \quad (8)$$

adequates the vigor chains of $\kappa(t)$, out of the orders 1 , $\boldsymbol{\chi},\boldsymbol{\chi}^{\mathbf{2}},\ldots,\boldsymbol{\chi}^{\boldsymbol{M}+\boldsymbol{Z}}$. PAM offers a power gauge with force chains to execute computational jobs[35]. The main goal of this paper is to solve fractional physical models via the STPAM. The STPAM gains exact and approximate numerical solutions analytically. This is the main advantage of STPAM. The exact solution received via STM and the approximate numerical solution achieved via PAM The outputs show the advantages of STPAM, including its ease of use, effectiveness, universality, cleanliness, packability, quality, and clarity.The remaining portions of the document had organized as follows: Section 2 introduces the STPAM designers. Section 3 contains the fractional physical models. Section 4 of the paper presents the discssion. Section 5 includes the study's conclusion.

2. The methodology of STPAM

Here, we display the proceedings of STPAM as follows: Let the following fractional nonlinear equation:

$$D_t^{\varepsilon}\kappa + \Re\kappa + \perp \kappa = \aleph(\chi,t), n-1 < \varepsilon \leq n,$$
$$\varepsilon \in N,\ n \in N, \quad (9)$$

where $\Re, \perp \kappa$, are linear and nonlinear functions, consequently, $\aleph(\chi,t)$, is an radix function, and $\kappa_0(\chi,\tau) = \kappa(\chi,0)$, is the initial condition (IC).

Step 1:

If we stratify STM on (9), and utilize IC, we profit:

$$u^{-\varepsilon}S[\kappa(\chi,t)] - u^{-\varepsilon}\kappa(\chi,0)$$
$$+S[\Re\kappa + \perp \kappa] = S[\aleph(\chi,t)], \quad (10)$$

By simplifying, we profit

$$S[\kappa(\chi,t)] = \kappa(\chi,0) - u^{\varepsilon}S[\Re\kappa + \perp \kappa]$$
$$+u^{\varepsilon}S[\aleph(\chi,t)], \quad (11)$$

Employing ADM [9,10], we can mark a nonlinear function$\perp \kappa(\chi,t) = \sum_{r=0}^{\infty}\Phi_r$, where

$$\Phi_r = \frac{1}{r!}\frac{d^r}{d\lambda^r}[\perp \sum_{r=0}^{\infty}\lambda^r \kappa_r]_{\lambda=0}, r = 0,1,2,\ldots, \quad (12)$$

Stratify the inverse ST on (11), we profit

$$\kappa(\chi,0) = \kappa_0(\chi,t), q \geq 0\ \kappa_{q+1}(\chi,t)$$
$$= S^{-1}[-u^{\varepsilon}S[\Re\kappa + \perp \kappa] + u^{\varepsilon}S[\aleph(\chi,t)]], , \quad (13)$$

The fractional solution takes the form:

$$\kappa(\chi,t) = \kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \ldots, \quad (14)$$

i.e.,

$$\kappa(\chi,t) = \kappa_0 + c_1\frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + c_2\frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}$$
$$+c_3\frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + c_4\frac{t^{4\varepsilon}}{\Gamma(4\varepsilon+1)} + \ldots, \quad (15)$$

Eq. (15) displays the fractional sequence solutions to (9) profited via STM. Lastly, STM supplies a closed-form solution, making it simple, attractive, and superior to competing approaches.

Step 2:

With great precision, we create a fractional numerical PAM solution to (9). Employing the oblique fractional Pade' approximation, manipulate:

$$\left[\frac{M}{\eta}\right] = \frac{a_0+a_1t^{\varepsilon}+a_2t^{2\varepsilon}+\ldots\ldots+a_Mt^{M\varepsilon}}{1+b_1t^{\varepsilon}+b_2\tau^{2\varepsilon}+\ldots\ldots+b_Z\tau^{\eta\varepsilon}}, \quad (16)$$

where $\mathrm{M} = \eta$,

Thus,

$$\left[\frac{\mathrm{M}}{\eta}\right] = \frac{a_0 + a_1t^{\varepsilon} + a_2t^{2\varepsilon} + \cdots \ldots + a_Mt^{\mathrm{M}\varepsilon}}{1 + b_1t^{\varepsilon} + b_2t^{2\varepsilon} + \cdots \ldots + b_Zt^{\eta\varepsilon}}$$
$$= \kappa_0 + c_1\frac{\tau^{\varepsilon}}{\Gamma(\varepsilon+1)} + c_2\frac{\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + \ldots, \quad (17)$$

By cross-multiplication in (17), we find

$$a_0 + a_1t^{\varepsilon} + a_2t^{2\varepsilon} + \ldots + a_Mt^{\mathrm{M}\varepsilon}$$
$$= c_0 + (c_0b_1 + c_1\frac{1}{\Gamma(\varepsilon+1)})t^{\varepsilon} + ((c_2 +$$
$$b_2c_0)\frac{1}{\Gamma(2\varepsilon+1)} + c_1b_1\frac{1}{\Gamma(\varepsilon+1)})t^{2\varepsilon} + \ldots, \quad (18)$$

In the case of powers of t, we get:

Agent of t^0: $a_0 = c_0$,

Agent of t^{ε}: $a_1 = (c_0b_1 + c_1\frac{1}{\Gamma(\varepsilon+1)})$,…, (19)

Notice that grade of $t^{\mathrm{M}\varepsilon+1}$,$t^{\mathrm{M}\varepsilon+2}$,…, $t^{2\mathrm{M}\varepsilon}$, have to be equalized to zero. Such straighten the stables in (16) via the Mathematica software.

3. Physical Models

Here, we investigate the solutions of the fractional physical model to clarify the advantages of the STPAM.

3.1 Model 1:

This model teaches the physical problem of wave propagation. The wave equation describes the vibrations of a string, the propagation of electromagnetic and sound waves, or the transmission of electric signals in a cable. The lessons of the wave equation are necessary in diverse areas of science and engineering. Consider the following wave equation [35]:

$$\frac{\mathrm{d}\kappa}{\mathrm{d}t} = \kappa_{\chi\chi}(\chi,t), 0 < \chi < \pi, t > 0. \quad (20)$$

We can write the fractional from of (20) as :

$$\frac{d^{\varepsilon}\kappa}{dt^{\varepsilon}} = \kappa_{\chi\chi}(\chi,t), 1 < \varepsilon \leq 2, 0 < \chi < \pi, t > 0, \quad (21)$$

with initial conditions

$$\kappa(\chi,0) = \sin\chi, \kappa_t(0,t) = 0, \quad (22)$$

At $\varepsilon \to 2$, we rest to the exact solution which is the same solution of (20) obtained by [35]:

$$\kappa(\chi,\mathrm{t}) = \sin\chi\cos t, \quad (23)$$

Stratifying ST on (21), we profit:

$$u^{-\varepsilon}S[k(\chi,t)] - u^{-\varepsilon}\kappa(\chi,0)$$
$$-u^{-\varepsilon-1}\kappa_t(0,t) = S[\kappa_{\chi\chi}], \quad (24)$$

On spreading, we profit

$S[(\chi,t)] = sin\,\chi + u^{\varepsilon}S[\kappa_{\chi\chi}],$ (25)

Stratify the invers STof (25), we profit:

$\kappa(\chi,t) = sin\,\chi + S^{-1}[u^{\varepsilon}S[\kappa_{\chi\chi}]],$ (26)

Employing the decomposition series :

$\kappa(\chi,t) = \sum_{q=0}^{\infty}\kappa_q(\chi,t)$, we profit

$\sum_{q=0}^{\infty}\kappa_q(\chi,t) = sin\,\chi + S^{-1}[u^{\varepsilon}S[\sum_{q=0}^{\infty}k_{qxx}(\chi,t)]],$ (27)

Employing ADM [9,10] and profit:

$\kappa(\chi,0) = sin\,\chi, \kappa_{q+1} = S^{-1}[u^{\varepsilon}S[\kappa_{q\chi\chi}]],$ (28)

i.e,

$$\kappa_1 = -\frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)}sin\,\chi, \kappa_2 = \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)}sin\,\chi,$$
$$\kappa_3 = -\frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)}sin\,\chi, \kappa_4 = \frac{t^{4\varepsilon}}{\Gamma(4\varepsilon+1)}sin\,\chi,$$
$$\kappa_5 = -\frac{t^{5\varepsilon}}{\Gamma(5\varepsilon+1)}sin\,\chi, \kappa_6 = \frac{t^{6\varepsilon}}{\Gamma(6\varepsilon+1)}sin\,\chi, \quad (29)$$

Then, the fractional solution of (21) is

$\kappa(\chi,t) = \kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \ldots.$ (30)

Thus,

$$\kappa(\chi,t) = \boldsymbol{sin}\,\chi\,[1 - \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + \frac{t^{4\varepsilon}}{\Gamma(4\varepsilon+1)} - \frac{t^{5\varepsilon}}{\Gamma(5\varepsilon+1)} + \ldots], \quad (31)$$

At $\varepsilon \to 2$, (31) becomes

$$\kappa(\chi,t) = sin\,\chi\,[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \ldots]. \quad (32)$$

i.e. $\kappa(\chi,t) = \mathrm{sinx}\sum_{n=0}^{\infty}\frac{-1^n}{(2n)!}t^{2n}$ (33)

Then, $\kappa(\chi,\tau) = sin\,\chi\,cos\,t,$ (34)

this is the exact solution of (21) which is the similar solution of (20) profited via [35]. The approximate solution of (21) for various values of ε can profit using PM $\left[\frac{3}{3}\right]$ as follows:

$$\left[\frac{3}{3}\right] = \frac{a_0 + a_1t^{\varepsilon} + a_2t^{2\varepsilon} + a_3t^{3\varepsilon}}{1 + b_1t^{\varepsilon} + b_2t^{2\varepsilon} + b_3t^{3\varepsilon}}$$
$$= sin\,\chi \rightleftarrows [1 - \frac{t^{\varepsilon}}{\Gamma(\varepsilon+1)} + \frac{t^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - \frac{t^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + \frac{t^{4\varepsilon}}{\Gamma(4\varepsilon+1)} - \frac{t^{5\varepsilon}}{\Gamma(5\varepsilon+1)} + \frac{t^{6\varepsilon}}{\Gamma(6\varepsilon+1)} - \ldots], \quad (35)$$

Cross multiplying yields and equating powers of t leads to:

Agent of $t^{6\varepsilon}$:

$$sin\,\chi\,(\frac{1}{\Gamma(6\varepsilon+1)} - b_1\frac{1}{\Gamma(5\varepsilon+1)} + b_2\frac{1}{\Gamma(4\varepsilon+1)} - b_3\frac{1}{\Gamma(3\varepsilon+1)}) = 0,$$

Agent of $t^{5\varepsilon}$:

$$sin\,\chi\,(-\frac{1}{\Gamma(5\varepsilon+1)} + b_1\frac{1}{\Gamma(4\varepsilon+1)} - b_2\frac{1}{\Gamma(3\varepsilon+1)} + b_3\frac{1}{\Gamma(2\varepsilon+1)}) = 0,$$

Agent of $t^{4\varepsilon}$:

$$sin\,\chi\,(\frac{1}{\Gamma(4\varepsilon+1)} - b_1\frac{1}{\Gamma(3\varepsilon+1)} + b_2\frac{1}{\Gamma(2\varepsilon+1)} - b_3\frac{1}{\Gamma(\varepsilon+1)}) = 0,$$

Agent of $t^{3\varepsilon}$:

$$sin\,\chi\,(-\frac{1}{\Gamma(3\varepsilon+1)} + b_1\frac{1}{\Gamma(2\varepsilon+1)} - b_2\frac{1}{\Gamma(\varepsilon+1)} + b_3) = a_3,$$

Agent of $t^{2\varepsilon}$:

$$sin\,\chi\,(\frac{1}{\Gamma(2\varepsilon+1)} - b_1\frac{1}{\Gamma(\varepsilon+1)} + b_2) = a_2,$$

Agent of t^{ε}: $sin\,\chi\,(-\frac{1}{\Gamma(\varepsilon+1)} + b_1\frac{1}{\Gamma(\varepsilon+1)}) = a_1,$

Agent of t^0: $a_0 = sin\,\chi,$

this system of equations gives the same fractional from of $b_1, b_2, b_3, a_0, a_1, a_2, a_3$ using Mathematica software. At $\varepsilon \to 2$, the previous system gives:

$b_1 = 0.0294042, b_2 = 0.000423729, b_3 = 3.23555x10^{-6},$

$a_0 = sin\,\chi, a_1 = -0.485298\,sin\,\chi, a_2 = 0.0273883\,sin\,\chi,$

$a_3 = -0.000372342\,sin\,\chi$

Then,

$$\left[\frac{3}{3}\right]_{PAM} = sin\,\chi\,\frac{1 - 0.485298t^2 + 0.0273883t^4 - 0.000372342t^6}{1 + 0.0294042t^2 + 0.000423729t^4 + 0.00000323554t^6} \quad (36)$$

Table 1: Absolute errors of $\kappa(\chi,t)$, via PAM versus the standard solution at $\varepsilon \to 2$, $x = 90$.

t	Exact solution	PAM	errorPAM
0.2	0.876176	0.875651	0.000525
0.4	0.823425	0.821332	0.002093
0.6	0.737847	0.733165	0.004682
0.8	0.622853	0.614598	0.008255
1.0	0.483028	0.470265	0.012763

Table 2: Absolute errors of $\kappa(\chi,t)$, via STM versus the standard solution at $\varepsilon \to 2$, $x = 90$.

t	Exact solution	STM	ErrorSTM
0.2	0.876176	0.876176	0.00000
0.4	0.823425	0.823425	0.00000
0.6	0.737847	0.737847	0.00000
0.8	0.622853	0.622853	0.00000
1.0	0.483028	0.483028	0.00000

Consider the following heat conduction [35]:

$$\frac{d\kappa}{dt} = \kappa_{\chi\chi}(\chi,y,z,t) + \kappa_{yy}(\chi,y,z,t) + \kappa_{zz}(\chi,y,z,t),$$
$$0 < \chi, y, z < \pi, t > 0. \quad (37)$$

Here, we write the fractional heat equation as follows:

$$\frac{d^{\varepsilon}\kappa}{dt^{\varepsilon}} = \kappa_{\chi\chi}(\chi,y,z,t) + \kappa_{yy}(\chi,y,z,t) + \kappa_{zz}(\chi,y,z,t),$$
$$0 < \varepsilon \le 1, 0 < \chi, y, z < \pi, t > 0, \quad (38)$$

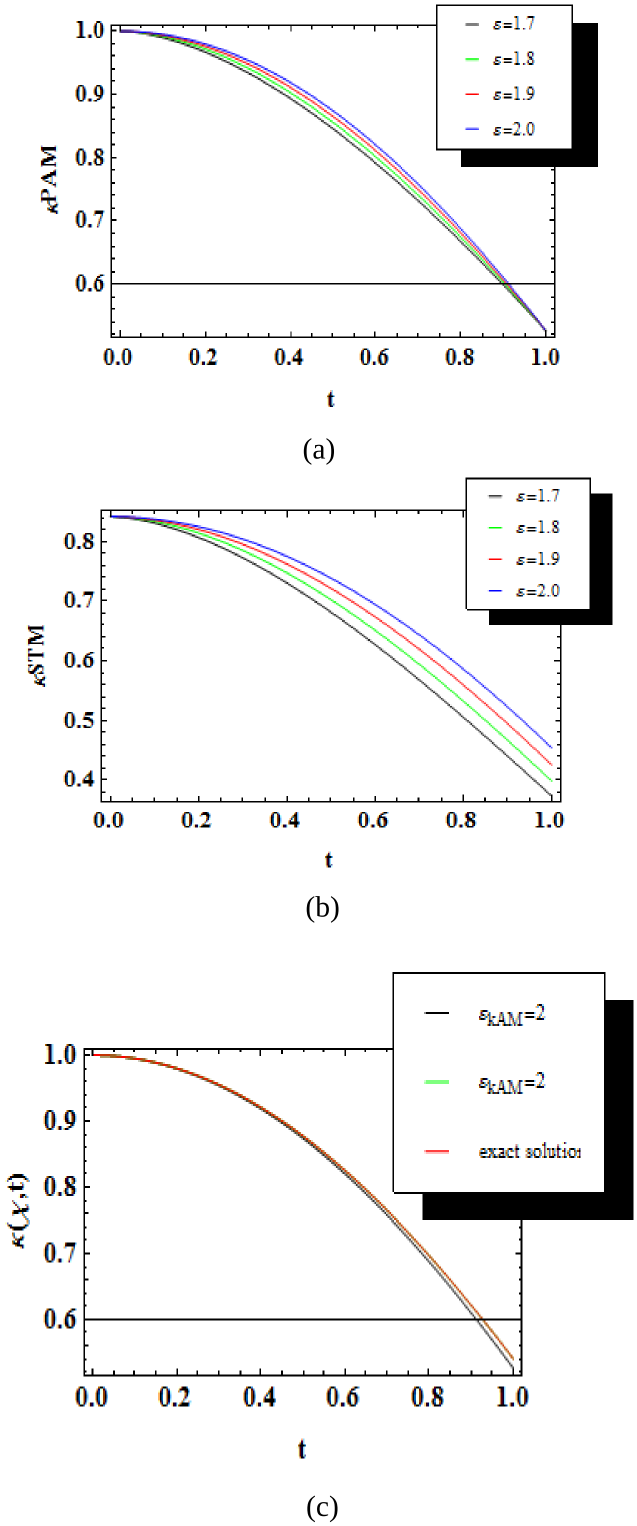


Figure 1: The plot of $\kappa(\chi,t)$, at different values of ε(a) PAM, (b) STM, and (c) the exact solution of (21) at $\varepsilon \to 2$, which is the same solution of (20) gained via [35].

with initial conditions:

$$\kappa(\chi,y,z,0) = 2\sin\chi \sin y \sin z \quad (39)$$

At $\varepsilon \to 1$, we rest to the exact solution $\kappa(\chi,y,z,t) = 2e^{-3t}\sin\chi\sin y\sin z$, of (37) which is the same solution of (36) obtained via [35]. Applying the ST on (37), we profit

$$u^{-\varepsilon}S[\kappa(\chi,y,z,t)] - u^{-\varepsilon}\kappa(\chi,y,z,0) = S[\kappa_{\chi\chi} + \kappa_{yy} + \kappa_{zz}], \quad (40)$$

On simplifying, we profit

$$S[\kappa(\chi,y,z,t)] = 2\,\mathbf{sin}\,\chi\,\mathbf{sin}\,y\,\mathbf{sin}\,z + u^{\varepsilon}S[\kappa_{\chi\chi} + \kappa_{yy} + \kappa_{zz}], \quad (41)$$

Taking the invers ST of (40), we profit

$$\kappa(\chi,y,z,t) = 2\,\mathbf{sin}\,\chi\,\mathbf{sin}\,y\,\mathbf{sin}\,z + S^{-1}[u^{\varepsilon}S[\kappa_{\chi\chi} + \kappa_{yy} + \kappa_{zz}]], \quad (42)$$

Employing the decomposition series $\kappa(\chi,y,z,t) = \sum_{q=0}^{\infty}\kappa_q(\chi,y,z,t)$, we profit

$$\sum_{q=0}^{\infty}\kappa_q(\chi,y,z,t) = 2\sin\chi\sin y\sin z + S^{-1}[u^{\varepsilon}S[\sum_{q=0}^{\infty}k_{q\chi\chi}(\chi,y,z,t) + \sum_{q=0}^{\infty}\kappa_{qyy}(\chi,y,z,t) + \sum_{q=0}^{\infty}\kappa_{qzz}(\chi,y,z,t)]], \quad (43)$$

Employing ADM [9,10] , we profit:

$$\kappa(\chi,y,z,0) = 2\,\mathbf{sin}\,\chi\,\mathbf{sin}\,y\,\mathbf{sin}\,z\,,$$
$$\kappa_{q+1} = S^{-1}[u^{\varepsilon}S[\kappa_{q\chi\chi} + \kappa_{qyy} + \kappa_{qzz}]]. \quad (44)$$

i.e,

$$\kappa_1 = -\frac{3t^{\varepsilon}}{\Gamma(\varepsilon+1)}2\,\mathbf{sin}\,\chi\,\mathbf{sin}\,y\,\mathbf{sin}\,z,$$
$$\kappa_2 = \frac{(3t)^{2\varepsilon}}{\Gamma(2\varepsilon+1)}2\,\mathbf{sin}\,\chi\,\mathbf{sin}\,y\,\mathbf{sin}\,z\,,$$
$$\kappa_3 = -\frac{(3t)^{3\varepsilon}}{\Gamma(3\varepsilon+1)}2\,\mathbf{sin}\,\chi\,\mathbf{sin}\,y\,\mathbf{sin}\,z,$$
$$\kappa_4 = \frac{(3t)^{4\varepsilon}}{\Gamma(4\varepsilon+1)}2\,\mathbf{sin}\,\chi\,\mathbf{sin}\,y\,\mathbf{sin}\,z\,,$$
$$\kappa_5 = -\frac{(3t)^{5\varepsilon}}{\Gamma(5\varepsilon+1)}2\,\mathbf{sin}\,\chi\,\mathbf{sin}\,y\,\mathbf{sin}\,z,$$
$$\kappa_6 = \frac{(3t)^{6\varepsilon}}{\Gamma(6\varepsilon+1)}2\,\mathbf{sin}\,\chi\,\mathbf{sin}\,y\,\mathbf{sin}\,z, \quad (45)$$

Then, the fractional solution of (38) is

$$\kappa(\chi,y,z,t) = \kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \ldots. \quad (46)$$

Thus,

$$\kappa(\chi,y,z,t) = 2\,\mathbf{sin}\,\chi\,\mathbf{sin}\,y\,\mathbf{sin}\,z \times [1 - \frac{3t^{\varepsilon}}{\Gamma(\varepsilon+1)} + \frac{(3t)^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - \frac{(3t)^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + \frac{(3t)^{4\varepsilon}}{\Gamma(4\varepsilon+1)} - \frac{(3t)^{5\varepsilon}}{\Gamma(5\varepsilon+1)} + \frac{(3t)^{6\varepsilon}}{\Gamma(6\varepsilon+1)} - \ldots], \quad (47)$$

At $\varepsilon \to 1$, (47) becomes

$$\kappa(\chi,y,z,t) = 2\,\mathbf{sin}\,\chi\,\mathbf{sin}\,y\,\mathbf{sin}\,z\,[1 - \frac{3t}{1!} + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \frac{(3t)^4}{4!} - \ldots], \quad (48)$$

Then,

$$\kappa(\chi,y,z,t) = 2e^{-3t}\sin\chi\sin y\sin z \quad (49)$$

The numerical approximate solution of (38) for various values of ε can obtain using PAM $\left[\frac{3}{3}\right]$ as follows:

$$\left[\frac{3}{3}\right] = \frac{a_0 + a_1 t^{\varepsilon} + a_2 t^{2\varepsilon} + a_3 t^{3\varepsilon}}{1 + b_1 t^{\varepsilon} + b_2 t^{2\varepsilon} + b_3 t^{3\varepsilon}}$$
$$= 2\sin\chi \sin y \sin z \rightleftarrows [1 - \frac{3t^{\varepsilon}}{\Gamma(\varepsilon+1)} + \frac{(3t)^{2\varepsilon}}{\Gamma(2\varepsilon+1)} - \frac{(3t)^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + \frac{(3t)^{4\varepsilon}}{\Gamma(4\varepsilon+1)} - \frac{(3t)^{5\varepsilon}}{\Gamma(5\varepsilon+1)} + \frac{(3t)^{6\varepsilon}}{\Gamma(6\varepsilon+1)} - \ldots], \quad (50)$$

cross multiplying yields and equating powers of t leads to:

Agent of $t^{6\varepsilon}$:

$$2\sin\chi \sin y \sin z \left(\frac{3^6}{\Gamma(6\varepsilon+1)} - b_1 \frac{3^5}{\Gamma(5\varepsilon+1)} + b_2 \frac{3^4}{\Gamma(4\varepsilon+1)} - b_3 \frac{3^3}{\Gamma(3\varepsilon+1)}\right) = 0,$$

Agent of $t^{5\varepsilon}$:

$$2\sin\chi \sin y \sin z \left(-\frac{3^5}{\Gamma(5\varepsilon+1)} + b_1 \frac{3^4}{\Gamma(4\varepsilon+1)} - b_2 \frac{3^3}{\Gamma(3\varepsilon+1)} + b_3 \frac{3^2}{\Gamma(2\varepsilon+1)}\right) = 0,$$

Agent of $t^{4\varepsilon}$:

$$2\sin\chi \sin y \sin z \left(\frac{3^4}{\Gamma(4\varepsilon+1)} - b_1 \frac{3^3}{\Gamma(3\varepsilon+1)} + b_2 \frac{3^2}{\Gamma(2\varepsilon+1)} - b_3 \frac{3}{\Gamma(\varepsilon+1)}\right) = 0,$$

Agent of $t^{3\varepsilon}$:

$$2\sin\chi \sin y \sin z \left(-\frac{3^3}{\Gamma(3\varepsilon+1)} + b_1 \frac{3^2}{\Gamma(2\varepsilon+1)} - b_2 \frac{3}{\Gamma(\varepsilon+1)} + b_3\right) = a_3,$$

Agent of t^{ε}:

$$2\sin\chi \sin y \sin z \left(-\frac{3}{\Gamma(\varepsilon+1)} + b_1\right) = a_1,$$

Agent of t^0:

$a_0 = 2\sin\chi \sin y \sin z,$

This system gives the same fractional from of $b_1, b_2, b_3, a_0, a_1, a_2, a_3$, using Mathematica software. At $\varepsilon \to 1$, we obtain:

$b_1 = 1.5, b_2 = 0.9, b_3 = 0.225,$

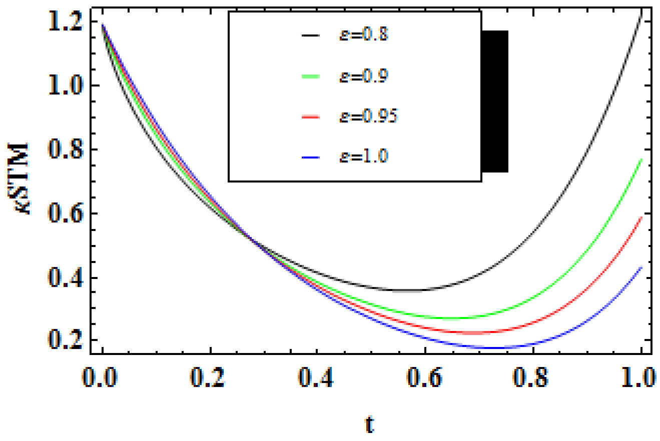
$a_0 = 2\sin x \sin y \sin z, a_1 = (-1.78747)2\sin x \sin y \sin z,$

$a_2 = (1.07248)2\sin x \sin y \sin y,$

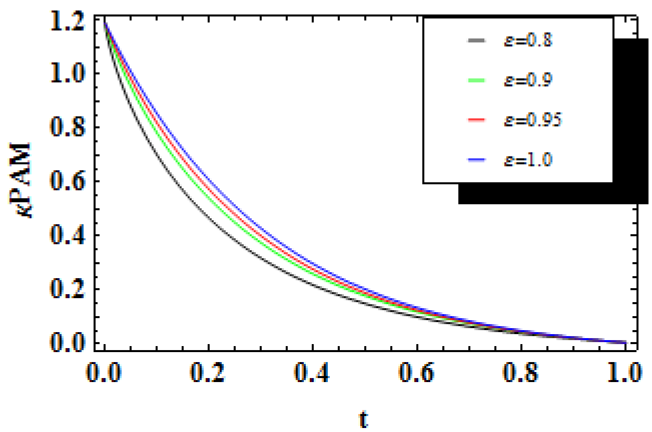
$a_3 = (-0.26812)2\sin x \sin y \sin z,$

Then,

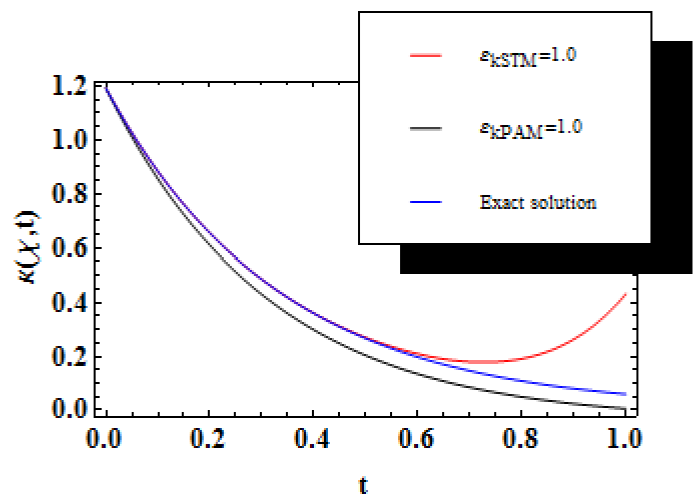
$$\left[\frac{3}{3}\right]_{PAM} = 2\sin\chi \sin y \sin z \frac{1 - 1.78747t + 1.07248t^2 - 0.26812t^3}{1 + 1.5t + 0.9t^2 + 0.225t^3} \quad (51)$$



(a)



(b)



(c)

Figure 2: Plot of $\kappa(\chi, y, z, t)$, at different values of $\mathcal{E}$ using (a) STM, (b) PAM, and (c) the exact solution of (38) at $\varepsilon \to 1$, which is the same solution gained via [35].

Table 3: Absolute errors of $\kappa(\chi, y, z, t)$, via PAM versus the standard solution at $\varepsilon \to 1$, $\chi, y, z = 90$.

t	Exact solution	PAM	errorPAM
0.2	1.16998	0.729717	0.440263
0.4	0.957899	0.356742	0.601157
0.6	0.784262	0.160326	0.623936
0.8	0.642099	0.058572	0.583527
1.0	0.525706	0.006604	0.519103

Table 4: Absolute errors of $\kappa(\chi, y, z, t)$, via STM versus the standard solution at $\varepsilon \to 1$, $\chi, y, z = 90$.

t	Exact solution	STM	ErrorSTM
0.2	1.16998	1.16998	0.00000
0.4	0.957899	0.957899	0.00000
0.6	0.784262	0.784262	0.00000
0.8	0.642099	0.642099	0.00000
1.0	0.525706	0.525706	0.00000

4. Discussion and conculsion:

To obtain adequate new rational solutions for the fractional physical models based on a combination of the ST and PAM approaches, this work proposes a novel technique dubbed STPAM. The computations had performed using the Mathematica software. Figures (**1,2**) and tables(**1-4**) show that the ST solution contrasts with an exact solution and a fractional solution created using Pade' approximation method (PAM).The current solutions are general solutions of the proposed models concerning standard cases produced by [35].On the other side. The STPAM differs from existing methods in that it is easy for manufacturers to use, doesn't demand arduous work, and provides superior alternatives. The treatment of fractional physical models using STPAM is a theoretical examination of models. It combined STM and PAM techniques. The STM is a method of analysis. The PAM can broaden the ingathering of the Taylor chains that had their ends severed and improve the rough ingathering average of the Maclaurin series. The computations were handled by Mathematica software. The results reveal the effectiveness and high quality of the current technique. Without employing linearization, perturbation, or constraining assumptions, the proposal technique presents the solutions as convergent series with readily calculable components. Consequently, the STPAM is effective and powerful in locating approximate, analytical-numerical solutions.

Data availability statement

The data used to support the findings of this study are available from the corresponding author upon request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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