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Characterization of Phani Distribution Based on Generalized Order Statistics

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Abstract: This paper focuses on the derivation of recurrence relations for both single and product moments of generalized order statistics derived from the Phani distribution. Special attention is given to order statistics and records as specific cases. Additionally, the characterization of the Phani distribution is achieved by utilizing recurrence relations for single and product moments.
Keywords: Generalized order statistics - Order statistics – Records –Single and product moments – Recurrence relations - Phani distribution - Characterization.

1. Introduction

The probability density function (pdf) of a random sample X is said to have a Phani distribution (PD) if it is in the form:

$$f(x)=\frac{\alpha\beta\theta}{(\theta-x)^2}\left(\frac{x}{\theta-x}\right)^{\beta-1}e^{-\alpha\left(\frac{x}{\theta-x}\right)^{\beta}};\ 0<x<\theta \qquad (1)$$

The cumulative distribution function (CDF) and survival function (SF) are:

$$F(x)=1-e^{-\alpha\left(\frac{x}{\theta-x}\right)^{\beta}};\qquad 0<x<\theta, \qquad (2)$$

and

$$\bar{F}(x)=e^{-\alpha\left(\frac{x}{\theta-x}\right)^{\beta}};\qquad 0<x<\theta. \qquad (3)$$

Substituting from Eq.(3) in Eq.(1), we get

$$f(x)=\frac{\alpha\beta\theta}{(\theta-x)^2}\left(\frac{t}{\theta-x}\right)^{\beta-1}\bar{F}(x)$$

$$\bar{F}(x)=\frac{f(x)(\theta-x)^2}{\alpha\beta\theta}\left(\frac{t}{\theta-x}\right)^{1-\beta}$$

Using the following binomial expansions

$$\bar{F}(x)=\frac{1}{\alpha\beta}\sum_{\nu=0}^{\infty}\theta^{\beta-\nu}\binom{\beta+1}{\nu}(-1)^{\nu}x^{\nu+1-\beta}f(x) \qquad (14)$$

Another name of this distribution is Wiebull-Uniform distribution introduced by Bourguignon et al. [1]. Modi and Gill [2] provide additional information on this distribution and its applications.

Kamps [3] introduced the concept of generalised order statistics (gos). This concept encompasses a wide range of order models for random variables.

For the sake of simplicity, let F denote an absolutely continuous distribution function f with density function f throughout. If the random variables $X(1,n,\tilde{m},k),\ldots,$ $X(n,n,\tilde{m},k)$ have a joint pdf of the form, they are called generalised order statistics (gos) based on F.

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[\bar{F}(x_i)\right]^{m_i}f(x_i)\right)f(x_n)\left[\bar{F}(x_n)\right]^{k-1},$$

for $F^{-1}(0)<x_n\le\ldots\le x_2\le x_1<F^{-1}(1)$, with parameters $n\in N,\ n\ge 2,\ k>0,$

$M_r=\sum_{i=r}^{n-1}m_i, \tilde{m}=(m_1,m_2,\ldots,m_{n-1})\in R^{n-1},$ such that $\gamma_r=k+n-r+M_r>0,$ for all $r\in\{1,\ 2,\ \ldots,\ n-1\}$.

For $\gamma_i\ne\gamma_j,\ j\ne i$ for all $j,i\in(1,2,\ldots,n)$ Cramer and Kamps [4] gave the pdf of $X(r,n,\tilde{m},k)$ by:

$$f_{X(r,n,\tilde{m},k)}(x)=f(x)C_{r-1}\sum_{i=1}^{r}\left[\bar{F}(x)\right]^{\gamma_i-1}a_i(r) \qquad (5)$$

The joint pdf of $X(s,n,\tilde{m},k)$ and , $X(r,n,\tilde{m},k)$ is given as $1\le r<s\le n$

$$f_{X(s,n,\tilde{m},k),X(r,n,\tilde{m},k)}(x,y)= C_{s-1}\left(\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\right)\left(\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i}\right)\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}, \qquad (6)$$

where $x<y$ and

$$a_i(r)=\prod_{\substack{j=1\\ j\neq i}}^{r}\frac{1}{\gamma_j-\gamma_i},\quad 1\le i\le r\le n,$$

$$a_i^{(r)}(s)=\prod_{\substack{j=r+1\\ j\neq i}}^{s}\frac{1}{\gamma_j-\gamma_i},\quad r+1\le i\le s\le n.$$

It needs to be pointed out that for $m_1=m_2=...=m_{n-1}=m\neq -1$,

$$a_i(r)=\frac{(-1)^{r-i}}{(m+1)^{r-1}(r-1)!}\binom{r-1}{r-i}, \tag{7}$$

and

$$a_i^{(r)}(s)=\frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!}\binom{s-r-1}{s-i}. \tag{8}$$

Consequently, the pdf of $X(r,n,\tilde{m},k)$ in Eq.(5) prevents to

$$f_{X(r,n,m,k)}(x)=\frac{C_{r-1}}{(r-1)!}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right], \tag{9}$$

and joint pdf of $X(s,n,\tilde{m},k)$ and $X(r,n,\tilde{m},k)$ given in Eq.(6) reduces to

$$f_{X(s,n,m,k),X(r,n,m,k)}(x,y)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\left[\bar{F}(x)\right]^{m}f(x)g_m^{r-1}\left[F(x)\right]\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s-1}f(y),\quad x<y \tag{10}$$

where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i,\quad \gamma_i=k+(n-i)(m+1),$$

$$h_m(x)=\begin{cases}\frac{-1}{m+1}(1-x)^{m+1}, & m\neq -1\\ -\ln(1-x), & m=-1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(0),\quad x\in[0,\ 1).$$

Also, taking $X(0,n,m,k)=0$. If k=1, m = 0, then $X(r,n,m,k)$ reduces to the $(n-r+1)^{th}$ order statistics, $X_{n-r+1:n}$ from the sample $X_1,X_2,...,X_n$ and when m = -1, then $X(r,n,m,k)$ reduces to the kth record values (Pawlas and Szynal [5]).The following are the rth generalised TL-moments with t_1 smallest and t_2 largest trimming:

$$L_r^{(t_1,t_2)}=\frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r-k+t_1:r+t_1+t_2});$$

$$t_1,t_2=1,2,\ldots \text{ and } r=1,2,\ldots, \tag{11}$$

where $E\left(X_{r-i+t_1:r+t_1+t_2}\right)$ is the expected value of the $(r-i+t_1)$th order statistics of the random sample of size $(r+t_1+t_2)$. The case $t_1=t_2=0$ yields the original L-moments defined by [6,7]).

The characterization has been extensively utilized by numerous researchers in their respective works, encompassing multiple studies. [8-19]

This paper presents the characterization of PD through the utilization of a recurrence relation for both single and product moments of generalized order statistics (GOS).

2. Recurrence relation for single Expectations of GOS

The single moments of generalised order statistics (GOS) for PD are calculated in this section. Special cases such as moments of order statistics, TL-moments, and L-moments can also be obtained from single GOS moments. Recurrence relations for GOS single moments are also shown. The resulting expressions represent single GOS moments for PD.

$$E\left[X^j(r,n,\tilde{m},k)\right]=\frac{C_{r-1}}{(r-1)!}\int_0^{\infty}x^j\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$

$$=\frac{C_{r-1}}{(m+1)^{r-1}(r-1)!}\int_0^{\infty}x^j\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)\left[1-\left(\bar{F}(x)\right)^{m+1}\right]^{r-1}dx$$

$$=\frac{C_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^w}{(m+1)^{r-1}(r-1)!}\int_0^{\infty}x^j\left[\bar{F}(x)\right]^{\gamma_r+w(m+1)-1}f(x)dx$$

$$=\frac{jC_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^w}{(m+1)^{r-1}(r-1)![\gamma_r+w(m+1)]}\int_0^{\infty}x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r+w(m+1)}dx$$

using Eq.(3), we get

$$E\left[T^j(r,n,\tilde{m},k)\right]=\frac{jC_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^w}{(m+1)^{r-1}(r-1)![\gamma_r+w(m+1)]}\int_0^{\theta}t^{j-1}\left[e^{-\alpha\left(\frac{t}{\theta-t}\right)^{\beta}}\right]^{\gamma_r+w(m+1)}dt \tag{12}$$

$$let\ a=\alpha\left[\gamma_r+w(m+1)\right]$$

$$I_1=\int_0^{\theta}t^{j-1}\left[e^{-\alpha\left(\frac{t}{\theta-t}\right)^{\beta}}\right]^{\gamma_r+w(m+1)}dt=\int_0^{\theta}t^{j-1}e^{-a\left(\frac{t}{\theta-t}\right)^{\beta}}dt$$

First, to obtain I_1, the binomial expansion is employed as follows

$$I_1=\sum_{\delta=0}^{\infty}\frac{(-1)^{\delta}a^{\delta}}{\delta!}\int_0^{\theta}t^{j-1}\left(\frac{t}{\theta-t}\right)^{\beta\delta}dt$$

$$=\sum_{\delta=0}^{\infty}\frac{(-1)^{\delta}a^{\delta}}{\delta!}\int_0^{\theta}t^{j-1+\beta\delta}(\theta-t)^{-\beta\delta}dt \quad (13)$$

$$I_1=\sum_{\delta=0}^{\infty}\frac{(-1)^{\delta}a^{\delta}}{\delta!}\int_0^{\theta}t^{j-1}\left(\frac{\theta}{t}-1\right)^{-\beta\delta}dt$$

$$let\ \frac{1}{y}=\frac{\theta}{t}\Rightarrow t=\theta y\Rightarrow dt=\theta dy$$

$$I_1=\sum_{\delta=0}^{\infty}\frac{(-1)^{\delta}a^{\delta}}{\delta!}\int_0^{1}(\theta y)^{j-1}\left(\frac{1}{y}-1\right)^{-\beta\delta}\theta dy$$

$$=\theta^{j}\sum_{\delta=0}^{\infty}\frac{(-1)^{\delta}a^{\delta}}{\delta!}\int_0^{1}y^{j-1}\left(\frac{1-y}{y}\right)^{-\beta\delta}dy$$

$$=\theta^{j}\sum_{\delta=0}^{\infty}\frac{(-1)^{\delta}a^{\delta}}{\delta!}\int_0^{1}y^{\beta\delta+j-1}(1-y)^{-\beta\delta}dy$$

So, I_1 is given by

$$I_1=\theta^{j}\sum_{\delta=0}^{\infty}\frac{(-1)^{\delta}a^{\delta}}{\delta!}\frac{\Gamma(\beta\delta+j)\Gamma(1-\beta\delta)}{\Gamma(\beta\delta+j+1)}$$

From Eq.(12), The single moments of gos for PD are

$$E\left[T^{j}(r,n,\tilde{m},k)\right]=jC_{r-1}\theta^{j}\sum_{w=0}^{r-1}\sum_{\delta}^{\infty}\binom{r-1}{w}(-1)^{w+\delta}$$
$$\frac{\alpha^{\delta}\left[\gamma_r+w(m+1)\right]^{\delta-1}}{\delta!(m+1)^{r-1}(r-1)!}\frac{\Gamma(\beta\delta+j)\Gamma(1-\beta\delta)}{\Gamma(\beta\delta+j+1)} \quad (14)$$

This represents the mathematical expression for the single moments of gos derived from the PD.

2.1. Moments of Upper Order Statistics

The single moments of generalised order statistics (GOS) for PD are calculated using equation Eq.(14). In addition, numerical calculations are performed to determine the mean and variance of upper order statistics for different parameter values.

The j^{th} moment of Upper order statistics is found by putting k=1, m=0 in Eq.(16) as

$$E\left(T^{j}_{n-r+1:n}\right)=jn!\theta^{j}\sum_{w=0}^{r-1}\sum_{\delta}^{\infty}\binom{r-1}{w}(-1)^{w+\delta}$$
$$\frac{\alpha^{\delta}\left[\gamma_r+w(m+1)\right]^{\delta-1}}{\delta!(n-r)!(r-1)!}\frac{\Gamma(\beta\delta+j)\Gamma(1-\beta\delta)}{\Gamma(\beta\delta+j+1)}$$

Or, by substituting $n-r+1=r$, the $E\left(T^{j}_{n-r+1:n}\right)$ will be $E\left(T^{j}_{r:n}\right)$ and takes the following form

$$E\left(T^{j}_{r:n}\right)=\frac{jn!\theta^{j}\sum_{w=0}^{r-1}\sum_{\delta}^{\infty}\binom{r-1}{w}}{\delta!(n-r)!(r-1)!}$$
$$(-1)^{w+\delta}\alpha^{\delta}\left[\gamma_r+w(m+1)\right]^{\delta-1} \quad (15)$$
$$\frac{\Gamma(\beta\delta+j)\Gamma(1-\beta\delta)}{\Gamma(\beta\delta+j+1)}$$

Tables 2.1 and 2.2 show the computed mean and variance of order statistics for the PD, which correspond to different parameter values.

Table 2.1: Mean of order statistics.

N	r	$\alpha=2$, $\beta=1$, $\theta=4$	$\alpha=1$, $\beta=2$, $\theta=3$	$\alpha=3$, $\beta=2$, $\theta=4$	$\alpha=5$, $\beta=6$, $\theta=8$
1	1	1.109	1.313	1.273	3.288
2	1	0.698	1.082	1.008	3.069
	2	1.52	1.544	1.539	3.507
3	1	0.514	0.955	0.870	2.944
	2	1.068	1.337	1.284	3.32
	3	1.746	1.648	1.666	3.601
4	1	0.407	0.869	0.780	2.856
	2	0.833	1.212	1.139	3.207
	3	1.303	1.462	1.429	3.434
	4	1.894	1.710	1.745	3.656
5	1	0.337	0.806	0.715	2.789
	2	0.685	1.123	1.04	3.125
	3	1.055	1.346	1.289	3.330
	4	1.469	1.540	1.523	3.503
	5	2	1.752	1.80	3.695
6	1	0.228	0.756	0.665	2.734
	2	0.583	1.055	0.965	3.061
	3	0.890	1.260	1.189	3.254
	4	1.220	1.431	1.388	3.406
	5	1.594	1.594	1.591	3.551
	6	2.084	1.784	1.842	3.724
7	1	0.252	0.715	0.625	2.689
	2	0.508	0.999	0.906	3.008
	3	0.771	1.193	1.113	3.193
	4	1.048	1.350	1.291	3.335
	5	1.348	1.492	1.461	3.460
	6	1.692	1.635	1.642	3.587
	7	2.146	1.809	1.876	3.746

It is worth noting that the results in Table 2.1 are consistent with the property of order statistics $\sum_{i=1}^{n}\mu_{i:n}=n\mu_{1:1}$ presented by David and Nagaraja [20].

For example: based on Table 2.1

$$\sum_{i=1}^{2}\mu_{1:2}=0.698+1.52=2.218,$$

and,

$$2\mu_{1:1}=2\times1.109=2.218,$$

then $\sum_{i=1}^{2}\mu_{1:2}=2\mu_{1:1}$, which justify this property.

Table 2.2: Variance of order statistics

n	r	$\alpha=2$, $\beta=1$, $\theta=4$	$\alpha=1$, $\beta=2$, $\theta=3$	$\alpha=3$, $\beta=2$, $\theta=4$	$\alpha=5$, $\beta=6$, $\theta=8$
1	1	0.519	0.167	0.217	0.159
2	1	0.275	0.140	0.159	0.150
	2	0.425	0.087	0.133	0.072
3	1	0.171	0.122	0.129	0.144
	2	0.280	0.080	0.106	0.067
	3	0.345	0.059	0.098	0.048
4	1	0.116	0.108	0.109	0.14
	2	0.198	0.074	0.091	0.065
	3	0.251	0.055	0.080	0.044
	4	0.289	0.045	0.079	0.037
5	1	0.085	0.098	0.095	0.137
	2	0.147	0.069	0.080	0.063
	3	0.191	0.052	0.070	0.042
	4	0.223	0.042	0.064	0.033
	5	0.249	0.037	0.067	0.031
6	1	0.064	0.09	0.084	0.134
	2	0.114	0.065	0.072	0.062
	3	0.151	0.049	0.063	0.041
	4	0.178	0.040	0.057	0.031
	5	0.198	0.034	0.054	0.027
	6	0.220	0.032	0.059	0.027
7	1	0.050	0.083	0.076	0.131
	2	0.091	0.061	0.066	0.061
	3	0.122	0.047	0.058	0.040
	4	0.146	0.038	0.052	0.030
	5	0.164	0.032	0.048	0.025
	6	0.178	0.028	0.047	0.023
	7	0.197	0.028	0.053	0.024

2.2. TL Moments

In this subsection, the r^{th} TL- moment and r^{th} L- moment for the PD are obtained.

The r^{th} TL- moment can be obtained from (11) and (15) with $j=1$, $n=r+t_1+t_2$ and $n-r+1=r-k+t_1$ as follows:

$$E\left(T_{r-k+t_1:r+t_1+t_2}\right)=\theta^{j}\sum_{w=0}^{n-r+k-t_1}\sum_{\delta=0}^{\infty}\binom{n-r+k-t_1}{w}\frac{(-1)^{w+\delta}\left(r+t_1+t_2\right)!\alpha^{\delta}\left[\gamma_r+w(m+1)\right]^{\delta-1}\alpha^{\delta}}{\delta!\left(n-r+k-t_1\right)!\left(r-k+t_1-1\right)!}\frac{\Gamma(\beta\delta+j)\Gamma(1-\beta\delta)}{\Gamma(\beta\delta+j+1)}$$

Then, the r^{th} TL- moment of the PD is obtained by substituting the previous expectation in Eq.(11) as follows

Furthermore, the r^{th} L- moments can be obtained from Eq.(16) with $t_1=t_2=0$ as follows:

$$L_r^{(t_1,t_2)}=\theta^{r}\sum_{k=0}^{r-1}\frac{\left(r+t_1+t_2\right)!(-1)^{k}\binom{r-1}{k}}{\left(n-r+k-t_1\right)!\left(r-k+t_1-1\right)!}\sum_{w=0}^{n-r+k-t_1}\sum_{\delta=0}^{\infty}\binom{n-r+k-t_1}{w}\frac{\Gamma(\beta\delta+j)\Gamma(1-\beta\delta)}{\Gamma(\beta\delta+j+1)};$$
$$\frac{(-1)^{w+\delta}\left(r+t_1+t_2\right)!\alpha^{\delta}\left[\gamma_r+w(m+1)\right]^{\delta-1}}{\delta!\left(n-r+k-t_1\right)!\left(r-k+t_1-1\right)!} \quad (16)$$

$t_1,t_2=1,2,\ldots$

and $r=1,2,\ldots$

Furthermore, the r^{th} L- moments can be obtained from (16) with $t_1=t_2=0$ as follows:

$$L_r=\theta^{r}\sum_{k=0}^{r-1}\frac{r!(-1)^{k}\binom{r-1}{k}}{(n-r+k)!(r-k-1)!}\sum_{w=0}^{n-r+k}\sum_{\delta=0}^{\infty}\binom{n-r+k}{w}\frac{\Gamma(\beta\delta+j)\Gamma(1-\beta\delta)}{\Gamma(\beta\delta+j+1)}\frac{(-1)^{w+\delta}r!\alpha^{\delta}\left[\gamma_r+w(m+1)\right]^{\delta-1}}{\delta!(n-r+k)!(r-1)!} \quad (17)$$

The first four L-moments can be obtained from Eq.(17) by taking $r=1,2,3$ and 4 respectively. Using Eq.(16), some numerical results for $L_1^{(t_1,t_2)},L_2^{(t_1,t_2)},L_3^{(t_1,t_2)},L_4^{(t_1,t_2)},L_1,L_2,L_3,L_4,\tau_1^{(s,t)},\tau_3^{(s,t)},\tau_4^{(s,t)},\tau_1,\tau_2$ and τ_3 are obtain in Table (2.3)

Using Eq. (15), some numerical results for mean and variance of order statistics are obtained in Athter, et al. [21].

(t_1,t_2)	$L_1^{(t_1,t_2)}$	$L_2^{(t_1,t_2)}$	$L_3^{(t_1,t_2)}$	$L_4^{(t_1,t_2)}$	$\tau_1^{(t_1,t_2)}$	$\tau_3^{(t_1,t_2)}$	$\tau_4^{(t_1,t_2)}$
$\alpha=2$ $\beta=1$ $\theta=4$							
(1,1)	1.068	0.235	0.015	0.0055			
(2,2)	1.055	0.165	0.0078	0.0023			
(0,1)	0.698	0.277	0.015	0.0058			
(0,2)	0.514	0.213	0.0071	0.0025	6.4	0.047	0.014
(1,0)	1.52	0.339	0.04	0.018			
(2.0)	1.746	0.295	0.039	0.017			
(0,0)	1.109	0.411	0.041	0.019			
$\alpha=1$ $\beta=2$ $\theta=3$							
(1,1)	1.337	0.125	-0.009	0.0067			
(2,2)	1.346	0.085	-0.005	0.0028			
(0,1)	1.082	0.191	-0.031	0.017			
(0,2)	0.955	0.171	-0.007	0.015	15.7	-0.059	0.034
(1,0)	1.544	0.155	-0.008	0.012			
(2.0)	1.648	0.124	0.0061	0.0086			
(0,0)	1.313	0.231	-0.024	0.023			
$\alpha=3$ $\beta=2$ $\theta=4$							
(1,1)	1.284	0.145	-0.005	0.007			
(2,2)	1.289	0.1	-0.003	0.003			
(0,1)	1.008	0.207	-0.023	0.015			
(0,2)	0.87	0.18	-0.025	0.013	12.9	-0.027	0.03

(1,0)	1.539	0.191	0.008	0.014			
(2.0)	1.666	0.158	0.014	0.012			
(0,0)	1.273	0.265	-0.011	0.024			
$\alpha = 5 \quad \beta = 6 \quad \theta = 8$							
(1,1)	3.32	0.113	-0.011	0.008	43.6	-0.07	0.043
(2,2)	3.33	0.076	-0.005	0.003			
(0,1)	3.069	0.188	-0.041	0.025			
(0,2)	2.944	0.176	-0.044	0.023			
(1,0)	3.507	0.14	-0.001	0.013			
(2.0)	3.601	0.111	0.006	0.008			
(0,0)	3.288	0.219	-0.032	0.03			

3.Characterization of PD utilized on single moments of gos

Theorem 2.1 Let X be a non-negative random variable with an absolutely continuous distribution function F (x) with F (0) = 0 and 0 < F(x) < 1 for all $x > 0$, then

$$E\left[X^j(r,n,\tilde{m},k)\right]-E\left[X^j(r-1,n,\tilde{m},k)\right]= \frac{j}{\alpha\beta\gamma_r}\sum_{i=0}^{\infty}\theta^{\beta-i}\binom{\beta+1}{i}(-1)^i E\left[X^{j+i-\beta}(r,n,\tilde{m},k)\right] \quad (18)$$

if and only if. $\bar{F}(x)=e^{-\alpha\left(\frac{x}{\theta-x}\right)^{\beta}}$

Proof

(i) The necessary part

We have from Lemma 2.3 (see [21]) that

$$E\left[\xi\{X(r,n,\tilde{m},k)\}\right]-E\left[\xi\{X(r-1,n,\tilde{m},k)\}\right]= C_{r-2}\sum_{i=1}^{r}a_i(r)\int_{\theta}^{\beta}\xi'(x)\left[\bar{F}(x)\right]^{\gamma_i}dx$$

If we let $\xi(x)=x^j$, then

$$E\left[X^j(r,n,\tilde{m},k)\right]-E\left[X^j(r-1,n,\tilde{m},k)\right]= jC_{r-2}\sum_{i=1}^{r}a_i(r)\int_{\theta}^{\beta}x^{j-1}\left[\bar{F}(x)\right]^{\gamma_i}dx \quad (19)$$

On using (4) in (12), we get

$$E\left[X^j(r,n,\tilde{m},k)\right]-E\left[X^j(r-1,n,\tilde{m},k)\right] =\frac{jC_{r-1}}{\alpha\beta\gamma_r}\sum_{i=0}^{\infty}\theta^{\beta-i}\binom{\beta+1}{i}(-1)^i \int_0^{\theta}x^{j+i-\beta}\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i-1}f(x)dx$$

Which after simplification leads to Eq.(18).

(ii) The sufficient part

Equation Eq.(5) yields the following result if the recurrence relation in equation Eq.(18) is satisfied:

$$\frac{C_{r-1}}{(r-1)!}\int_0^{\theta}x^j\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx- \frac{C_{r-2}}{(r-2)!}\int_0^{\theta}x^j\left[\bar{F}(x)\right]^{\gamma_{r-1}-1}f(x)g_m^{r-2}\left[F(x)\right]dx=$$

$$\frac{jC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\sum_{i=0}^{\infty}\theta^{\beta-i}\binom{\beta+1}{i}(-1)^i \int_0^{\theta}x^{j+i-\beta}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$

By applying integration by parts to the first term on the left-hand side, we obtain

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\theta}\left[\bar{F}(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]x^{j-1}dx= \frac{jC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\sum_{i=0}^{\infty}\theta^{\beta-i}\binom{\beta+1}{i}(-1)^i \int_0^{\theta}x^{j+i-\beta}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$

This is implies that

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\theta}x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right] \left\{\bar{F}(x)-\frac{1}{\alpha\beta}\sum_{i=0}^{\infty}\theta^{\beta-i}\binom{\beta+1}{i}(-1)^i x^{1+i-\beta}f(x)\right\}dx=0 \quad (20)$$

By employing a generalized version of the Muntz-Szasz theorem ([22]) to equation Eq.(20), we obtain

$$\bar{F}(x)=\frac{1}{\alpha\beta}\sum_{i=0}^{\infty}\theta^{\beta-i}\binom{\beta+1}{i}(-1)^i x^{i+1-\beta}f(x)$$

Hence,

$$\bar{F}(x)=\frac{\theta^{\beta}\left(1-\frac{x}{\theta}\right)^{\beta+1}x^{1-\beta}f(x)}{\alpha\beta}$$

$$\bar{F}(x)=\frac{\left(\frac{\theta}{x}-1\right)^{\beta+1}x^2 f(x)}{\alpha\theta\beta}$$

Integrating both side from 0 to y, we get

$$\int_0^{y}\frac{f(x)}{\bar{F}(x)}dx=\alpha\beta\theta\int_0^{y}\left(\frac{\theta}{x}-1\right)^{-\beta-1}x^{-2}dx$$

This is implies that

$$-\ln\left[\bar{F}(y)\right]=\alpha\left(\frac{\theta}{y}-1\right)^{-\beta}$$

Hence

$$\bar{F}(y)=e^{-\alpha\left(\frac{y}{\theta-y}\right)^{\beta}}.$$

Corollary 3.2. For $m_1=m_2=\ldots=m_{n-1}=m\neq-1,$ the recurrence relations for single moment of gos for PD is given as

$$E\left[X^{j}(r,n,m,k)\right]-E\left[X^{j}(r-1,n,m,k)\right]=$$
$$\frac{j}{\alpha\beta\gamma_{r}}\sum_{i=0}^{\infty}\theta^{\beta-i}\binom{\beta+1}{i}(-1)^{i}E\left[X^{j+i-\beta}(r,n,m,k)\right] \quad (21)$$

Proof. This can easy be deduced from Eq.(11) in view of the relation Eq.(7).

Remark 3.1 Putting $m=0,\ k=1$ in Theorem 2.1., we obtain recurrence relations for single moments of order statistics as

$$E\left(X_{r:n}^{j}\right)-E\left(X_{r-1:n}^{j}\right)=$$
$$\frac{j}{\alpha\beta(n-r+1)}\sum_{i=0}^{\infty}\theta^{\beta-i}\binom{\beta+1}{i}(-1)^{i}E\left(X_{r:n}^{j+i-\beta}\right) \quad (22)$$

Remark 3.2 Setting $m=-1,\ k=1$ in Theorem 2.1., we obtain the recurrence relations of upper record values as

$$E\left[X^{j}(r,n,-1,1)\right]-E\left[X^{j}(r-1,n,-1,1)\right]$$
$$=\frac{j}{k\alpha\beta}\sum_{i=0}^{\infty}\theta^{\beta-i}\binom{\beta+1}{i}(-1)^{i}E\left[X^{j+i-\beta}(r,n,-1,1)\right] \quad (23)$$

4.Characterization of PD based on product moments of gos

Theorem 2.1 Let X be a non-negative random variable having an absolutely continuous distribution function F (x) with F (0) = 0 and 0 < F (x) < 1 for all $xy>0$, then

$$E\left[X^{i}(r,n,m,k).X^{j}(s,n,m,k)\right]-E\left[X^{i}(r,n,m,k).X^{j}(s-1,n,m,k)\right]$$
$$=\frac{j}{\alpha\theta\gamma_{s}}\sum_{\nu=0}^{\infty}\theta^{\beta-\nu}\binom{\beta+1}{\nu}(-1)^{\nu}E\left[X^{i}(r,n,m,k)X^{j+\nu-\beta}(s,n,m,k)\right] \quad (24)$$

if and only if. $\bar{F}(x)=e^{-\alpha\left(\frac{x}{\theta-x}\right)^{\beta}}$

Proof

(i) The necessary part

We have from Lemma 3.2 ([21])that

$$E\left[\xi\{X(r,n,m,k).X(s,n,m,k)\}\right]-E\left[\xi\{X(r,n,m,k).X(s-1,n,m,k)\}\right]=\frac{C_{s-2}}{(r-1)!(s-r-1)!}$$
$$\int_{\theta}^{\beta}\int_{x}^{\beta}\frac{\partial}{\partial y}\xi(x,y)\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]\left[h_{m}(F(y))-h_{m}(F(x))\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_{s}}dydx$$

where $\xi(x,y)=\xi_{1}(x)\xi_{2}(y)$

If we let $\xi(x,y)=x^{i}y^{j}$, then

$$E\left[X^{i}(r,n,m,k).X^{j}(s,n,m,k)\right]-E\left[X^{i}(r,n,m,k).X^{j}(s-1,n,m,k)\right]=\frac{C_{s-2}}{(r-1)!(s-r-1)!}$$
$$\int_{\theta}^{\beta}\int_{x}^{\beta}x^{i}y^{j-1}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]\left[h_{m}(F(y))-h_{m}(F(x))\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_{s}}dydx$$

On using (4) , we get

$$E\left[X^{i}(r,n,m,k).X^{j}(s,n,m,k)\right]-E\left[X^{i}(r,n,m,k).X^{j}(s-1,n,m,k)\right]=$$
$$\frac{jC_{s-2}\sum_{\nu=0}^{\infty}\theta^{\beta-\nu}\binom{\beta+1}{\nu}(-1)^{\nu}}{\alpha\beta(r-1)!(s-r-1)!}$$
$$\int_{0}^{\infty}\int_{x}^{\infty}x^{i}y^{j+\nu-\beta}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]\left[h_{m}(F(y))-h_{m}(F(x))\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_{s}-1}f(y)dydx$$

Which after simplification leads to Eq.(24).

(ii) The sufficient part

If the recurrence relation in equation Eq.(24) is satisfied, then by using Eq.(10), we have

$$\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{0}^{\infty}\int_{x}^{\infty}x^{i}y^{j}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]\left\{h_{m}\left[F(y)\right]-h_{m}\left[F(x)\right]\right\}^{s-r-1}$$
$$\left[\bar{F}(y)\right]^{\gamma_{s}-1}f(y)dydx=\frac{C_{s-2}}{(r-1)!(s-r-2)!}$$
$$\int_{0}^{\infty}\int_{x}^{\infty}x^{i}y^{j}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]\left\{h_{m}\left[F(y)\right]-h_{m}\left[F(x)\right]\right\}^{s-r-2}\left[\bar{F}(y)\right]^{\gamma_{s-1}-1}$$
$$f(y)dydx=\frac{jC_{s-1}\sum_{\nu=0}^{\infty}\theta^{\beta-\nu}\binom{\beta+1}{\nu}(-1)^{\nu}}{\alpha\beta\gamma_{s}(r-1)!(s-r-1)!}$$
$$\int_{0}^{\infty}\int_{x}^{\infty}x^{i}y^{j+\nu-\beta}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]\left\{h_{m}\left[F(y)\right]-h_{m}\left[F(x)\right]\right\}^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_{s}-1}f(y)dydx$$

Integrating the first term in left hand side by parts, we get

$$\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}$$
$$\int_0^\theta\int_x^\theta x^i y^{j-1}\left[\bar{F}(x)\right]^m f(x) g_m^{r-1}\left[F(x)\right]$$
$$\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s} dydx$$
$$=\frac{jC_{s-1}\sum_{\nu=0}^{\infty}\theta^{\beta-\nu}\binom{\beta+1}{\nu}(-1)^{\nu}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_0^\theta\int_x^\theta x^i y^{j+\nu-\beta}\left[\bar{F}(x)\right]^m f(x)$$
$$g_m^{r-1}\left[F(x)\right]\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1} f(y)dydx$$

this is implies that

$$\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}$$
$$\int_0^\theta\int_x^\theta x^i y^{j-1}\left[\bar{F}(x)\right]^m f(x) g_m^{r-1}\left[F(x)\right]$$
$$\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s-1} \tag{25}$$
$$\left\{\bar{F}(y)-\frac{1}{\alpha\beta}\sum_{\nu=0}^{\infty}\theta^{\beta-\nu}\binom{\beta+1}{\nu}(-1)^{\nu} y^{\nu+1-\beta} f(y)\right\}$$
$$dydx=0$$

Now applying a generalization of the Muntz-Szasz theorem [22] to equation Eq.(25), we get

$$\bar{F}(y)=\frac{1}{\alpha\beta}\sum_{i=0}^{\infty}\theta^{\beta-i}\binom{\beta+1}{i}(-1)^{i} y^{i+1-\beta} f(y)$$

Hence,

$$\bar{F}(y)=\frac{\theta^{\beta}\left(1-\frac{y}{\theta}\right)^{\beta+1} y^{1-\beta} f(y)}{\alpha\beta}$$
$$\bar{F}(y)=\frac{\left(\frac{\theta}{y}-1\right)^{\beta+1} y^{2} f(y)}{\alpha\theta\beta}$$

Integrating both side from 0 to x, we get

$$\int_0^x \frac{f(y)}{\bar{F}(y)}dy=\alpha\beta\theta\int_0^x\left(\frac{\theta}{y}-1\right)^{-\beta-1} y^{-2}dy$$

This is implies that

$$-\ln\left[\bar{F}(x)\right]=\alpha\left(\frac{\theta}{x}-1\right)^{-\beta}$$

Hence

$$\bar{F}(x)=e^{-\alpha\left(\frac{x}{\theta-x}\right)^{\beta}}$$

Remark 4.1 Putting $m=0,\ k=1$ in Eq.(24), we obtain recurrence relations for product moments of order statistics as

$$E\left[X_{r,s:n}^{i,j}\right]-E\left[X_{r,s-1:n}^{i,j}\right]$$
$$=\frac{j}{\alpha\theta(n-s+1)}\sum_{\nu=0}^{\infty}\theta^{\beta-\nu}\binom{\beta+1}{\nu}(-1)^{\nu}E\left[X_{r,s:n}^{i,j+\nu-\beta}\right] \tag{26}$$

Remark 4.2 Setting $m=-1$ in Eq.(24), we obtain the recurrence relations for product moments of kth record values as

$$E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^j\right]-E\left[\left(X_r^{(k)}\right)^i\left(X_{s-1}^{(k)}\right)^j\right]$$
$$=\frac{j}{k\alpha\theta}\sum_{\nu=0}^{\infty}\theta^{\beta-\nu}\binom{\beta+1}{\nu}(-1)^{\nu}E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^{j+\nu-\beta}\right] \tag{27}$$

4. Conclusion

In this paper some recurrence relations for single moments of Upper generalized order statistics for Ph distribution are derived. Some special moments include the moment of Upper order statistics, L-moment and TL-moment are obtained. Numerical values for the mean and variance for the Ph distribution from Upper order statistics are calculated for some values of parameters. Furthermore, recurrence relations for product moments based on Upper generalized order statistics are discussed. for the Ph distribution. Finally, a characterization of the Ph distribution in terms of single and product moments.

CRediT authorship contribution statement:

Conceptualization: Ali A. A-Rahman and Ibrahim B. Abdul-Moniem.; methodology, Salwa M. Assar and Khater A. E. Gad; software, Salwa M. Assar and Khater A. E. Gad.; validation, Salwa M. Assar and Khater A. E. Gad.; formal analysis, Ali A. A-Rahman and Ibrahim B. Abdul-Moniem.; investigation, Ali A. A-Rahman and Ibrahim B. Abdul-Moniem.; resources, Salwa M. Assar and Khater A. E. Gad.; data curation, Salwa M. Assar and Khater A. E. Gad.; writing—original draft preparation, Khater A. E. Gad.; writing—review and editing, Khater A. E. Gad.; visualization, A. A-Rahman and Ibrahim B. Abdul-Moniem.; supervision, Ali A. A-Rahman and Ibrahim B. Abdul-Moniem and Salwa M. Assar.; project administration, A-Rahman and Ibrahim B. Abdul-Moniem.; funding acquisition, Salwa M. Assar and Khater A. E. Gad. All authors have read and agreed to the published version of the manuscript.

Data availability statement

The data used to support the findings of this study are available from the corresponding author upon request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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