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Certain approximation spaces using local functions via idealization

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Abstract:

In this paper, we use the notion of upper set $\overline{R}(A)$ to define the local function and closure operator $cl_R^*(A)$ in an ideal approximation space $(X,R,\mathscr{L})$. This, ideal approximation space $(X,R,\mathscr{L})$ based on an ideal $\mathscr{L}$ joined to the approximation space (X,R) are introduces as well. The approximation axioms $T_i,\ i=0,1,2$ are introduced in the approximation space and also in the ideal approximation spaces. Examples are given to explain the definitions . Connectedness in approximation spaces and ideal connectedness are introduced and the differences between them are explained.
keywords: Ideal approximation space; local function; separation axioms; connectedness.

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The concept of rough sets was originated by Pawlak in 1982 [1] based on there are some objects in a vague area called the boundary region that can not be determined by a set or its complement. Rough sets depends on a relation R defined on the universal finite set X, and the pair (X,R) is called an approximation space. Firstly, rough sets was given by some equivalence relation. Many authors studied rough sets based on more generalized relations on X, for example see [2–5]. There are lower set, upper set and consequently a boundary region that became an essential role in artificial intelligence, granular computing and decision analysis. The generated topology τ on an approximation space (X,R) that represent the topological properties of rough sets were studied by many authors (ex. [2, 6–10]). Many kinds of generalizations of Pawlak's rough set were obtained by replacing the equivalence relation with an arbitrary binary relation. On the other hand, many researchers have studied the relationships between rough sets and topological spaces and have used topological structures like infra-topology and supra-topology to deal with rough set notions and address some real-life problems (ex. [8, 11–13]). It was proved that the lower and upper approximation operators derived by a reflexive and transitive relation were exactly the interior and closure operators in a topology. Many research works were introduced fore the ordinary case with rough sets with some medical applications as in[14–16].

Based on the paper in [17], if we combined the definitions given in [18] and the definitions given in [4] that used an ideal on X, then we get a more general form of roughness and a better accuracy value of the rough set. Thus, assigning an ideal in defining the lower and upper sets in some approximation space is a generalization of roughness.

In this paper, we introduce the interior and closure in ideal approximation spaces, generating two ideal approximation topological spaces based on minimal neighborhoods. The local functions of some subset (A) of a universe (X) with respect to a given ideal play a basic role in defining the related interior and closure operators. Separation axioms with respect to these ideal approximation spaces are reformulated and compared with examples to show their implications. We reformulate and study connectedness in these ideal approximation spaces and compare them with examples to show the implications between them. Ideal approximation and continuity are introduced. Moreover, we modified our definitions to get similar types of ideal approximation spaces but based on maximal neighborhoods. In addition, we explained the relationship between some of the topological properties of the two types with some examples.

In the course of the paper, let X be a finite set of objects as a universal set. A relation R from a universe X to a universe X (a relation on X) is a subset of $X \times X$. The formula $(x,y) \in R$ is abbreviated as xRy and means that x is in relation R with y. Also, the right neighborhoods of $x \in X$ is $xR = \{y : xRy\}$ and the left neighborhoods of $x \in X$ is $Rx = \{y : yRx\}$. A set $<x>R$ (resp. $R<x>$) is the intersection of all right (resp. left) neighborhoods containing x. Also, $R<x>R = R<x> \cap <x>R$. For $A \subseteq X$, the lower $(L_R(A))$, the upper $(U_R(A))$ and the boundary region $B_R(A)$ ate approximation sets defined as follows (see [3, 19])

$$L_R(A) = \{x \in A \ :<x>R \subseteq A\},$$
$$U_R(A) = A \cup \{x \in X \ :<x>R \cap A \neq \phi\},$$
$$B_R(A) = U_R(A) - L_R(A).$$

$L_R(A), U_R(A)$ and $B_R(A)$ are the called lower, upper and boundary region approximation sets associated with the set $A \subseteq X$ and based on $<x>R$ in an approximation space (X,R).

Lemma 01[20] *The upper approximation $U_R(A)$ has the following properties:*

(1)$U_R(\phi) = \phi$,
(2)$L_R(A) \subseteq A \subseteq U_R(A)$, for $A \subseteq X$,
(3)$U_R(A \cup B) = U_R(A) \cup U_R(B)$, $\forall A, B \subseteq X$,
(4)$U_R(U_R(A)) = U_R(A)$, $\forall A \subseteq X$,
(5)$U_R(A) = (L_R(A^c))^c$, $\forall A \subseteq X$, where A^c denotes the complement of A.

Also, the operator $U_R(A)$ on $P(X)$ induced a topology on X denoted by $\tau^{\sim}$ and defined by $\tau^{\sim} = \{A \subseteq X : U_R(A^c) = A^c\}$.

Definition 01[21] *Let X be a non-empty set. Then $\mathscr{L} \subseteq P(X)$ is called an ideal on X if it satisfies the following conditions:*

(1)$\phi \in \mathscr{L}$,
(2)If $A \in \mathscr{L}$ and $B \subseteq A$, then $B \in \mathscr{L}$,
(3)If $A, B \in \mathscr{L}$, then $A \cup B \in \mathscr{L}$.

Definition 02[4] *Let R be a binary relation on X and $\mathscr{L}$ be an ideal defined on X and $A \subseteq X$. Then, the lower and upper approximations, $\underline{R}(A)$ and $\overline{R}(A)$ of A are defined by:*

$$\underline{R}(A) = \{x \in A \ :<x>R \cap A^c \in \mathscr{L}\},$$
$$\overline{R}(A) = A \cup \{x \in X \ :<x>R \cap A \notin \mathscr{L}\}.$$

Lemma 02[4] *The upper approximation $\underline{R}(A)$ has the following properties:*

(1)$\overline{R}(A) = (\underline{R}(A^c))^c$,
(2)$\overline{R}(\phi) = \phi$,
(3)$\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$,
(4)If $A \subseteq B$, then $\overline{R}(A) \subseteq \overline{R}(B)$,
(5)$\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$,
(6)$\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$,
(7)$\overline{R}(\overline{R}(A)) = \overline{R}(A)$.

Definition 03[17] *Let R be a binary relation on X and $\mathscr{L}$ be an ideal on X and $A \subseteq X$. Then, the lower and upper approximations, $\underline{\underline{R}}(A)$ and $\overline{\overline{R}}(A)$ of A are defined by:*

$$\underline{\underline{R}}(A) = \{x \in A : R<x>R \cap A^c \in \mathscr{L}\},$$
$$\overline{\overline{R}}(A) = A \cup \{x \in X : R<x>R \cap A \notin \mathscr{L}\}.$$

Theorem 01[17] *The upper approximation $\overline{\overline{R}}(A)$ has the following properties: for $A, B \subseteq X$,*

(1)$\overline{\overline{R}}(A) = (\underline{\underline{R}}(A^c))^c$,
(2)$\overline{\overline{R}}(\phi) = \phi$,
(3)$L_R(A) \subseteq \underline{\underline{R}}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{\overline{R}}(A) \subseteq \overline{R}(A) \subseteq U_R(A)$,
(4)If $A \subseteq B$, then $\overline{\overline{R}}(A) \subseteq \overline{\overline{R}}(B)$,
(5)$\overline{\overline{R}}(A \cap B) \subseteq \overline{\overline{R}}(A) \cap \overline{\overline{R}}(B)$,
(6)$\overline{\overline{R}}(A \cup B) = \overline{\overline{R}}(A) \cup \overline{\overline{R}}(B)$,
(7)$\overline{\overline{R}}(\overline{\overline{R}}(A)) = \overline{R}(A)$.

1 Ideal approximation spaces

Definition 11*Let $(X, R, \mathscr{L})$ be any ideal approximation space and $A \subseteq X$. Then,*

(1)The $*-$local closed set A^* of A is defined by:

$$A^* = \bigcap\{G \subseteq X : A - G \in \mathscr{L}, \overline{R}(G) = G\}.$$

(2)The $**-$local closed set A^{**} of A is defined by:

$$A^{**} = \bigcap\{G \subseteq X : A - G \in \mathscr{L}, \overline{\overline{R}}(G) = G\}.$$

Corollary 11*Let $(X, R, \mathscr{L}_0)$ be any ideal approximation space, where $\mathscr{L}_0$ is the trivial ideal on X. Then, for each $A \subseteq X$ we have $A^* = \overline{R}(A)$ and $A^{**} = \overline{\overline{R}}(A)$.*

*Proof.*Since $\mathscr{L}_0 = \{\phi\}$, we get that $A^* = \bigcap\{G \subseteq X : A - G = \phi, \overline{R}(G) = G\}$, that is, $A^* = \bigcap\{G \subseteq X : A \subseteq G, \overline{R}(G) = G\}$. Since $A \subseteq \overline{R}(A), \overline{R}(\overline{R}(A)) = \overline{R}(A)$, then $A^* \subseteq \overline{R}(A)$. Suppose that $\overline{R}(A) \nsubseteq A^*$, then there exists $G \subseteq X, A \subseteq G, \overline{R}(G) = G$ so that $\overline{R}(A) \nsubseteq G = A^*$. But $A \subseteq G$ implies that $\overline{R}(A) \subseteq \overline{R}(G) = G$. Contradiction, and then, $A^* = \overline{R}(A)$. Similarly, $A^{**} = \overline{\overline{R}}(A)$.

Proposition 11*Let $(X, R, \mathscr{L})$ be any ideal approximation space,$\mathscr{L}_1$ and $\mathscr{L}_2$ be two ideals on X, and let A and B be subsets of X. Then the following properties hold:*

(1)$A \subseteq B$ implies $A^* \subseteq B^*$ and $A^{**} \subseteq B^{**}$,
(2)If $\mathscr{L}_1 \subseteq \mathscr{L}_2$, then $A^*(\mathscr{L}_1) \supseteq A^*(\mathscr{L}_2)$ and $A^{**}(\mathscr{L}_1) \supseteq A^{**}(\mathscr{L}_2)$,
(3)$A^* = \overline{R}(A^*) \subseteq \overline{R}(A)$ and $A^{**} = \overline{\overline{R}}(A^{**}) \subseteq \overline{\overline{R}}(A)$,
(4)$(A^*)^* = A^*$ and $(A^{**})^{**} = A^{**}$,
(5)$\underline{R}(A) \subseteq ((A^c)^*)^c$ and $\underline{R}(A) \subseteq (((A)^c)^{**})^c$,
(6)$(A \cap B)^* \subseteq A^* \cap B^*$ and $(A \cap B)^{**} \subseteq A^{**} \cap B^{**}$,

(7) $(A\cup B)^* = A^*\cup B^*$ *and* $(A\cup B)^{**} = A^{**}\cup B^{**}$,
(8) $A\in\mathscr{L}$, *if and only if* $A^{\sim}=\phi$, *and* $A\in\mathscr{L}$, *if and only if* $A^{**}=\phi$,
(9) $A^{**}\subseteq A^*$.

Proof. We proof for A^*, and A^{**} is by the same way.

(1) Suppose that $A^*\nsubseteq B^*$, then there exists $G\subseteq X$ with $B-G\in\mathscr{L}$ and $\overline{R}(G)=G$ such that $A^*\nsubseteq G=B^*$. Since $A\subseteq B$, then $A-G\subseteq B-G$ and $A-G\in\mathscr{L}$, $\overline{R}(G)=G$. Thus, $A^*\subseteq G$, which is a contradiction. Hence, $A^*\subseteq B^*$.
(2) Suppose that $A^*(\mathscr{L}_1)\nsupseteq A^*(\mathscr{L}_2)$, then there exists $G\subseteq X$ with $A-G\in\mathscr{L}_1$ and $\overline{R}(G)=G$ such that $A^*(\mathscr{L}_2)\nsubseteq G=A^*(\mathscr{L}_1)$. Since $\mathscr{L}_1\subseteq\mathscr{L}_2$, then $A-G\in\mathscr{L}_2$, $\overline{R}(G)=G$, and then $A^*(\mathscr{L}_2)\subseteq G$, which is a contradiction. Thus, $A^*(\mathscr{L}_1)\supseteq A^*(\mathscr{L}_2)$.
(3) It is obvious that, $\overline{R}(A^*)=A^*$ direct. Since $A\subseteq\overline{R}(A)$, $\overline{R}(\overline{R}(A))=\overline{R}(A)$, then $A^*\subseteq\overline{R}(A)$. Since $A^*\subseteq\overline{R}(A)$, then $\overline{R}(A^*)\subseteq\overline{R}(\overline{R}(A))=\overline{R}(A)$.
(4) From (3), we have $(A^*)^*=\overline{R}((A^*)^*)\subseteq\overline{R}(A^*)=A^*$. Hence, $(A^*)^*\subseteq A^*$.
Conversely, suppose that $A^*=K\nsubseteq(A^*)^*=G$. Then, $A^*-G\in\mathscr{L}$, $\overline{R}(G)=G$. and, $A-K\in\mathscr{L}$, $\overline{R}(K)=K$. Thus $K-G\in\mathscr{L}$, and $A-K\in\mathscr{L}$, then $A-G\subseteq(K-G)\cup(A-K)\in\mathscr{L}$, $\overline{R}(G)=G$, and therefore $A^*=K\subseteq G$, which is a contradiction. Hence $A^*=(A^*)^*$.
(5) From (3), we have $(A^c)^*\subseteq\overline{R}(A^c)$, then $\underline{R}(A)=(\overline{R}(A^c))^c\subseteq((A^c)^*)^c$.
(6) From (1), we have $(A\cap B)^*\subseteq A^*$ and $(A\cap B)^*\subseteq B^*$, then $(A\cap B)^*\subseteq A^*\cap B^*$.
(7) $A^*\cup B^*\subseteq(A\cup B)^*$ direct. Suppose that $(A\cup B)^*\nsubseteq A^*\cup B^*$, then there exists $G_1,G_2\subseteq X$ with $A-G_1\in\mathscr{L}$, $B-G_2\in\mathscr{L}$, $\overline{R}(G_1)=G_1$, $\overline{R}(G_2)=G_2$ such that $(A\cup B)^*\nsubseteq G_1\cup G_2$. Therefore $(A\cup B)-(G_1\cup G_2)\in\mathscr{L}$, $\overline{R}(G_1\cup G_2)=G_1\cup G_2$. Thus, $(A\cup B)^*\subseteq G_1\cup G_2$, which is a contradiction. Hence, $(A\cup B)^*\subseteq A^*\cup B^*$.
(8) If $A\in\mathscr{L}$, then $A-\phi=A\in\mathscr{L}$ and $\overline{R}(\phi)=\phi$. Hence $A^*=\phi$. Conversely, if $A^*=\phi$, then $A-\phi=A\in\mathscr{L}$.
(9) Suppose that $A^{**}\nsubseteq A^*$, then there exists $G\subseteq X$ with $A-G\in\mathscr{L}$ and $\overline{R}(G)=G$ such that $A^{**}\nsubseteq G=A^*$. Since $\overline{\overline{R}}(G)\subseteq\overline{R}(G)=G$, then $A^{**}\subseteq G$, which is a contradiction. Hence, $A^{**}\subseteq A^*$.

Remark 11 *Let $(X,R,\mathscr{L})$ be an ideal approximation space and $A,B\subseteq X$. The following examples show that in general:*

(1) $A^*\subseteq B^*\nRightarrow A\subseteq B$.
(2) $A^*(\mathscr{L}_1)\supseteq A^*(\mathscr{L}_2)\nRightarrow\mathscr{L}_1\subseteq\mathscr{L}_2$.
(3) $A^*\neq\overline{R}(A)$ *and* $\underline{R}(A)\neq((A^c)^*)^c$.
(4) $(A\cap B)^*\neq A^*\cap B^*$.
(5) $A^{**}\neq A^*$.

Example 11

(1) *Let* $X=\{x,y,z\}$, $R=\{(x,x),(x,y),(x,z),(y,y),(y,z),(z,z)\}$, *and* $\mathscr{L}=\{\phi,\{y\}\}$. *Then,* $<x>R=\{x,y,z\}$, $<y>R=\{y,z\}$, $<z>R=\{z\}$. *If* $A=\{y\}$, $B=\{x\}$, *then* $A^*=\phi$. *Also,* $\{x\}$ *is the smallest set with* $\{x\}-\{x\}=\phi\in\mathscr{L}$, $\overline{R}(\{x\})=\{x\}$, *then* $B^*=\{x\}$. *Thus,* $A^*\subseteq B^*$, *but* $A\nsubseteq B$.
(2) *In (1), if* $A=\{x\}$ *and* $\mathscr{L}_1=\{\phi,\{y\}\}$, $\mathscr{L}_2=\{\phi,\{x\}\}$, *then* $A^*(\mathscr{L}_1)=\{x\}\supseteq A^{\sim}(\mathscr{L})_2=\phi$, *but* $\mathscr{L}_1\nsubseteq\mathscr{L}_2$.
(3) *In (1),* $A=\{y\}$, $A^*=\phi$, *but* $\overline{R}(A)=\{y\}$. *So,* $A^*\neq\overline{R}(A)$. *Also, if* $A=\{x,z\}$, *then* $(A^c)^*=(\{y\})^*=\phi$. *But,* $\underline{R}(A)=\{x,z\}$. *Thus,* $\underline{R}(A)=\{x,z\}\neq((A^c)^*)^c=X$.
(4) *In (1), if* $\mathscr{L}=\{\phi,\{z\}\}$, $A=\{x\}$ *and* $B=\{y\}$, *then,* $A^*=\{x\}$ *and* $\{x,y\}$ *is the smallest set with* $\{y\}-\{x,y\}=\phi\in\mathscr{L}$, $\overline{R}(\{x,y\})=\{x,y\}$, *then* $B^*=\{x,y\}$ *but* $(A\cap B)^*=\phi^*=\phi$. *Thus,* $(A\cap B)^*\neq A^*\cap B^*$.
(5) *Let* $X=\{x,y,z\}=\{(x,x),(x,y),(x,z),(y,x),(y,y),(z,y),(z,z)\}$, $\mathscr{L}=\{\phi,\{x\}\}$ *Then,* $<x>R=\{x,y\}$, $<y>R=\{y\}$, $<z>R=\{y,z\}$. *Also,* $R<x>=\{x\}$, $R<y>=\{x,y\}$, $R<z>=\{x,z\}$. *Therefore,* $R<x>R=\{x\}$, $R<y>R=\{y\}$, $R<z>R=\{z\}$. *If* $A=\{x,y\}$, *then* X *is the smallest set with* $\{x,y\}-X=\phi\in\mathscr{L}$, $\overline{R}(X)=\overline{R}(X)=X$, *then* $A^*=X$. *Also,* $\{y\}$ *is the smallest set with* $\{x,y\}-\{y\}=\{x\}\in\mathscr{L}$, $\overline{\overline{R}}(\{y\})=\{y\}\cup\{x\in X:R<x>R\cap\{y\}\notin\mathscr{I}\}=\{y\}$, *then* $A^{**}=\{y\}$. *So,* $A^{**}\neq A^*$.

Definition 12 *Let $(X,R,\mathscr{L})$ be an ideal approximation space. Then, for any $A\subseteq X$, define the operators $int^*_R(A), cl^*_R(A), int^{**}_R(A), cl^{**}_R(A): P(X)\longrightarrow P(X)$ as follows:*

$$cl^*_R(A)=A\cup A^* \text{ and } int^*_R(A)=A\cap((A^c)^*)^c \quad \forall A\subseteq X \tag{1.1}$$

$$cl^{**}_R(A)=A\cup A^{**} \text{ and } int^{**}_R(A)=A\cap((A^c)^{**})^c \;\forall A\subseteq X. \tag{1.2}$$

Now, if $\mathscr{L}=\{\phi\}$, then from Corollary 11, $cl^*_R(A)=\overline{R}(A)=A^*$, $int^*_R(A)=\underline{R}(A)=((A^c)^*)^c$. and $cl^{**}_R(A)=\overline{\overline{R}}(A)=A^{**}$, $int^{**}_R(A)=\underline{\underline{R}}(A)=((A^c)^{**})^c$.

Proposition 12 *Let $(X,R,\mathscr{L})$ be an ideal approximation space. Then, for any $A,B\subseteq X$, we have:*

(1) $L_R(A)\subseteq int^*_R(A)\subseteq int^{**}_R(A)\subseteq A\subseteq cl^{**}_R(A)\subseteq cl^*_R(A)\subseteq U_R(A)$.
(2) $cl^*_R(A^c)=(int^*_R(A))^c$, $int^*_R(A^c)=(cl^*_R(A))^c$ *and* $cl^{**}_R(A^c)=(int^{**}_R(A))^c$, $int^{**}_R(A^c)=(cl^{**}_R(A))^c$.
(3) *If* $A\subseteq B$, *then* $cl^*_R(A)\subseteq cl^*_R(B)$, $int^*_R(A)\subseteq int^*_R(B)$ *and* $cl^{**}_R(A)\subseteq cl^{**}_R(B)$, $int^{**}_R(A)\subseteq int^{**}_R(B)$.
(4) $int^*_R(A\cap B)=int^*_R(A)\cap int^*_R(A)$ *and* $int^{**}_R(A\cap B)=int^{**}_R(A)\cap int^{**}_R(A)$.
(5) $int^*_R(A\cup B)\supseteq int^*_R(A)\cup int^*_R(A)$ *and* $int^{**}_R(A\cup B)\supseteq int^{**}_R(A)\cup int^{**}_R(A)$.
(6) $cl^*_R(A\cap B)\subseteq cl^*_R(A)\cap cl^*_R(B)$ *and* $cl^{**}_R(A\cap B)\subseteq cl^{**}_R(A)\cap cl^{**}_R(B)$.
(7) $cl^*_R(A\cup B)=cl^*_R(A)\cup cl^*_R(B)$ *and* $cl^{**}_R(A\cup B)=cl^{**}_R(A)\cup cl^{**}_R(B)$.
(8) $cl^*_R(cl^*_R(A))=cl^*_R(A)$ *and* $int^*_R(int^*_R(A))=int^*_R(A)$.
(9) $cl^{**}_R(cl^{**}_R(A))=cl^{**}_R(A)$ *and* $int^{**}_R(int^{**}_R(A))=int^{**}_R(A)$.

Proof. (1)Direct from Proposition 11 (3),(5),(9).
(2)

$$\begin{aligned}[int_R^*(A)]^c &= [A\cap((A^c)^*)^c]^c\\ &= (A^c\cup A^c)^*\\ &= cl_R^*(A^c).\end{aligned}$$

By the same way, we can prove that $int_R^*(A^c) = (cl_R^*(A))^c$.
(3)From Proposition 11 (1), we get $A\subseteq B$, implies that $cl_R^*(A) = A\cup A^*\subseteq B\cup B^* = cl_R^*(B)$. Second part is similar.
(4)By (2), we have

$$\begin{aligned}int_R^*(A\cap B) &= [cl_R^*(A\cap B)^c]^c\\ &= [cl_R^*(A^c\cup B^c)]^c\\ &= [cl_R^*(A^c)\cup cl_R^*(B^c)]^c\\ &= [cl_R^*(A^c)]^c\cap[cl_R^*(B^c)]^c\\ &= int_R^*(A)\cap int_R^*(A).\end{aligned}$$

Second part is similar.
(5)Similar to (4).
(6)From Proposition 11 (6), we get $cl_R^*(A\cap B) = (A\cap B)\cup(A\cap B)^*\subseteq(A\cap B)\cup(A^*\cap B^*)\subseteq(A\cup A^*)\cap(B\cup B^*) = cl_R^*(A)\cap cl_R^*(B)$.
(7)Similar to (6) by using Proposition 11 (7).
(8)From Proposition 11 (4), we get

$$\begin{aligned}cl_R^*(cl_R^*(A)) &= cl_R^*[A\cup A^*]\\ &= cl_R^*(A)\cup cl_R^*(A^*)\\ &= cl_R^*(A).\end{aligned}$$

Second part is similar.
(9)Similar to (8).

Corollary 12*Let $(X,R,\mathscr{L})$ be an ideal approximation space. Then, the operator $cl_R^*(A)$ on $P(X)$ defined in Equation 1.1, induces a topology on X denoted by τ^* and defined by $\tau^* = \{A\subseteq X : cl_R^*(A^c) = A^c\}$. Also, the operator $cl_R^{**}(A)$ on $P(X)$ defined in Equation 1.2, induces a topology on X denoted by τ^{**} and defined by $\tau^{**} = \{A\subseteq X : cl_R^{**}(A^c) = A^c\}$. It is clear that $\tau^{\sim}\subseteq\tau^*\subseteq\tau^{**}$.*

2 Lower separation axioms in ideal approximation spaces

Definition 21

(1)An ideal approximation space $(X,R,\mathscr{L})$ is T_0^* (resp. T_0^{**}) space iff $\forall x\neq y\in X$ there exists $A\subseteq X$ such that $x\in int_R^*(A)$, (resp. $x\in int_R^{**}(A)$) such that $y\notin A$ or $y\in int_R^*(A)$ (resp. $y\in int_R^{**}(A)$) such that $x\notin A$.
(2)An ideal approximation space $(X,R,\mathscr{L})$ is T_1^* (resp. T_1^{**}) space iff $\forall x\neq y\in X$ there exists $A,B\subseteq X$ such that $x\in int_R^*(A), y\in int_R^*(B)$ (resp. $x\in int_R^{**}(A), y\in int_R^{**}(B)$) such that $x\notin B$ and $y\notin A$.
(3)An ideal approximation space $(X,R,\mathscr{L})$ is T_2^* (resp. T_2^{**}) space iff $\forall x\neq y\in X$ there exists $A,B\subseteq X$ such that $x\in int_R^*(A), y\in int_R^*(B)$ (resp. $x\in int_R^{**}(A), y\in int_R^{**}(B)$) such that $A\cap B=\phi$.

Remark 21 *(1)From $L_R(A)\subseteq int_R^*(A)\subseteq int_R^{**}(A)$ we have the following implication.*

$$\begin{tikzcd}
T_2 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & T_1 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & T_o \arrow[d, Rightarrow] \\
T_2^* \arrow[r, Rightarrow] \arrow[d, Rightarrow] & T_1^* \arrow[r, Rightarrow] \arrow[d, Rightarrow] & T_o^* \arrow[d, Rightarrow] \\
T_2^{**} \arrow[r, Rightarrow] & T_1^{**} \arrow[r, Rightarrow] & T_o^{**}
\end{tikzcd}$$

Figure 1: Implication

(2)Consider an ideal approximation space $(X,R,\mathscr{L})$ and $\mathscr{L}=\{\phi\}$. Then, the ideal separation axioms T_i^* are identical to the separation axioms T_i, $i=0,1,2$.

Example 21

(1)Let $X=\{a,b,c\}, R=\{(a,a),(a,b),(b,b),(c,c)\}$. Then, $<a>R=\{a,b\}, <b>R=\{b\}, <c>R=\{c\}$. So, for $a\neq b, b\neq c$ there exists $\{b\}$ so that $b\in L_R(\{b\})=\{b\}$ and $a,c\notin\{b\}$. For $a\neq c$ there exists $\{c\}$ so that $c\in L_R(\{c\})=\{c\}$ and $a\notin\{c\}$. Hence, X is T_0-space but not T_1-space since we can not find a set $A\subseteq X$ such that $a\in L_R(A)$ and not containing b.
(2)In (1) if $\mathscr{L}=\{\phi,\{a\},\{c\},\{a,c\}\}$, then for $a\neq b, b\neq c$ there exists $A=\{b\}$ so that $(\{b\}^c)^*=(\{a,c\})^*=\phi$. Then, $int_R^*(A)=A\cap((A^c)^*)^c=\{b\}\cap X=\{b\}$ and $b\in\{b\}$ but $a,c\notin\{b\}$. For $a\neq c$ there exists $A=\{c\}$ so that $(A^c)^*=(\{a,b\})^*=\{a,b\}$. Then, $int_R^*(A)=A\cap((A^c)^*)^c=\{c\}\cap\{c\}=\{c\}$ and $c\in\{c\}$ but $a\notin\{c\}$. Hence, X is T_0^*-space. But X is not T_1^*-space since we can not find a set $A\subseteq X$ such that $a\in int_R^*(A)$ and not containing b.
(3)Let $X=\{a,b,c\}, R=\{(a,a),(a,b),(b,a),(b,b),(c,c)\}$ and $\mathscr{L}=\{\phi,\{a\},\{b\},\{a,b\}\}$, Then, $<a>R=\{a,b\}, <b>R=\{a,b\}, <c>R=\{c\}$. Then, there exist $A=\{a\}, B=\{b\}, C=\{c\}$ so that $(\{a\}^c)^*=(\{b,c\})^*=\{c\}$. Then, $int_R^*(A)=A\cap((A^c)^*)^c=\{a\}\cap\{a,b\}=\{a\}$. Also, $(\{b\}^c)^*=(\{a,c\})^*=\{c\}$. Then, $int_R^*(B)=B\cap((B^c)^*)^c=\{b\}\cap\{a,b\}=\{b\}$, and $(\{c\}^c)^*=(\{a,b\})^*=\phi$, $int_R^*(C)=C\cap((C^c)^*)^c=\{c\}\cap X=\{c\}$, which means for $a\neq b$, there exist $A,B\subseteq X$ such that

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$a \in int_R^*(A) = \{a\}, b \notin A$ and $b \in int_R^*(B) = \{b\}, a \notin \{b\}$. Similarly for $a \neq c$ and $b \neq c$.Hence, X is T_1^*-space. It is also T_0^*-space. But X is neither T_0-space nor T_1-space since we can not find a set $A \subseteq X$ such that $a \in L_R(A), b \notin A$ or $b \in L_R(A), a \notin A$.

(4)In (1) if $\mathscr{L} = \{\phi, \{a\}, \{b\}, \{a,b\}\}$, then $int_R^*(\{a\}) = \{a\}, int_R^*(\{b\}) = \{b\}$ and $int_R^*(\{c\}) = \{c\}$. Hence, X is T_2^*-space. But, X is not T_2-space since it is not T_1.

Example 22

(1)Let $X = \{a,b,c\}, R = \{(a,a),(a,b),(b,a),(b,b),(c,c)\}$ and $\mathscr{L} = \{\phi, \{b\}\}$. Then, $< a > R = \{a,b\}, < b > R = \{a,b\}, < c > R = \{c\}$. Also, $R < a >= \{a,b\}, R < b >= \{a,b\}, R < c >= \{c\}$. Therefore, $R < a > R = \{a,b\}, R < b > R = \{a,b\}, R < c > R = \{c\}$. Then for $a \neq b, a \neq c$ there exists $A = \{a\}$ so that $(\{a\}^c)^{**} = (\{b,c\})^{**} = \{c\}$. Then, $int_R^{**}(A) = A \cap ((A^c)^{**})^c = \{a\} \cap \{a,b\} = \{a\}$ and $a \in \{a\}$ but $b,c \notin \{a\}$. For $a \neq c$ there exists $A = \{c\}$ so that $(A^c)^{**} = (\{a,b\})^{**} = \{a,b\}$. Then, $int_R^{**}(A) = A \cap ((A^c)^{**})^c = \{c\} \cap \{c\} = \{c\}$ and $c \in \{c\}$ but $a \notin \{c\}$. Hence, X is T_0^{**}-space but not $T_1^{**}-$space since we can not find a set $A \subseteq X$ such that $b \in int_R^{**}(A)$ and not containing a.

(2)Let $X = \{a,b,c\}, R = \{(a,a),(a,b),(b,c),(c,c)\}$ and $\mathscr{L} = \{\phi, \{c\}\}$. Then, $< a > R = \{a,b\}, < b > R = \{a,b\}, < c > R = \{c\}$. Also, $R < a >= \{a\}, R < b >= \{b,c\}, R < c >= \{b,c\}$. Therefore, $R < a > R = \{a\}, R < b > R = \{b\}, R < c > R = \{c\}$.Then, there exist $A = \{a\}, B = \{b\}, C = \{c\}$ so that $(\{a\}^c)^{**} = (\{b,c\})^{**} = \{b\}$. Then, $int_R^{**}(A) = A \cap ((A^c)^{**})^c = \{a\} \cap \{a,c\} = \{a\}$. Also, $(\{b\}^c)^{**} = (\{a,c\})^{**} = \{a\}$. Then, $int_R^{**}(B) = B \cap ((B^c)^{**})^c = \{b\} \cap \{b,c\} = \{b\}$, and $(\{c\}^c)^{**} = (\{a,b\})^{**} = \{a,b\}$, $int_R^{**}(C) = C \cap ((C^c)^{**})^c = \{c\} \cap \{c\} = \{c\}$, which means for $a \neq b$, there exist $A, B \subseteq X$ such that $a \in int_R^{**}(A) = \{a\}, b \notin A$ and $b \in int_R^{**}(B) = \{b\}, a \notin \{b\}$. Similarly for $a \neq c$ and $b \neq c$. Hence, X is T_1^{**}-space. It is also T_0^{**}-space. But X is neither T_0^*-space nor T_1^*-space since we can not find a set $A \subseteq X$ such that $a \in int_R^*(A), b \notin A$ or $b \in int_R^*(A), a \notin A$.

(3)In (2) we have, $int_R^{**}(\{a\}) = \{a\}, int_R^{**}(\{b\}) = \{b\}$ and $int_R^{**}(\{c\}) = \{c\}$. Hence, X is $T_2^{**}-$space. But, X is not T_2^*-space since it is not T_1^*.

Example 23*Let X be an infinite set and $R = X \times X$. If $\mathscr{L}_f$ is an ideal of finite subsets of X, then*

$$\overline{R}(A) = \overline{\overline{R}}(A) = \begin{cases} A & \text{if } A \in \mathscr{L}_f \\ X & o.w. \end{cases}$$

Thus,

$$(A^c)^* = (A^c)^{**} = \begin{cases} \phi & \text{if } A^c \in \mathscr{L}_f \\ X & o.w. \end{cases}$$

and so,

$$int_R^*(A) = int_R^{**}(A) = \begin{cases} A & \text{if } A^c \in \mathscr{L}_f \\ \phi & o.w. \end{cases}$$

So, $\forall x \neq y \in X$, we have: $(x \in int_R^*(\{y\}^c) = int_R^{**}(\{y\}^c) = \{y\}^c, y \notin \{y\}^c)$ $(y \in int_R^*(\{x\}^c) = int_R^{**}(\{x\}^c) = \{x\}^c, x \notin \{x\}^c)$. Hence, X is T_1^*, T_1^{**}-space. But X is neither T_2^* nor T_2^{**}. space, since if $x \in int_R^*(A), y \in int_R^*R(B)$ and $A \cap B = \phi$, then $int_R^*(A) \cap int_R^*R(B) = \phi$ and $int_R^*R(A) \subseteq (int_R^*R(B))^c$ which is impossible because $int_R^*R(A)$ is infinite and $(int_R^*R(B))^c$ is finite.

Proposition 21*For an ideal approximation space $(X,R,\mathscr{I})$, the following are equivalent:*

(1)X is T_0^*-space.
(2)$cl_R^*(\{x\}) \neq cl_R^*(\{y\})$ for each $x \neq y \in X$.

Proof.

(1)$\Rightarrow$ (2): For each $x \neq y \in X$, by (1) there exists $A \subseteq X$ such that $x \in int_R^*(A) = A \cap ((A^c)^*)^c, y \notin A$. Then, $x \in ((A^c)^*)^c, y \in A^c$. Thus, $\{y\} - (A^c)^* \subseteq A^c - (A^c)^* \in \mathscr{L}, (A^c)^* = \overline{R}((A^c)^*)$. So,$(\{y\})^* \subseteq (A^c)^*$. Hence, $x \notin \{y\} \cup (\{y\})^* = cl_R^*(\{y\})$. Therefore,

$$cl_R^*(\{x\}) \neq cl_R^*(\{y\}).$$

(2)$\Rightarrow$ (1): For each $x \neq y \in X$, by (2), $x \notin cl_R^*(\{y\}) = \{y\} \cup (\{y\})^*$ or $y \notin cl_R^*(\{x\}) = \{x\} \cup (\{x\})^*$. Thus, $(x \in (((X - \{y\})^c)^*)^c, y \notin X - \{y\})$ or $(y \in (((X - \{x\})^c)^*)^c, x \notin X - \{x\})$, then $(x \in int_R^*(X - \{y\}), y \notin X - \{y\})$ or $(y \in int_R^*(X - \{y\}), x \notin X - \{x\})$, Hence $(Y,R,\mathscr{L})$ is T_0^*-space.

Corollary 21*For an ideal approximation space $(X,R,\mathscr{L})$, the following are equivalent:*

(1)X is $T_0^{**}-$space.
(2)$cl_R^{**}(\{x\}) \neq cl_R^{**}(\{y\})$ for each $x \neq y \in X$.

Proposition 22*For an ideal approximation space $(X,R,\mathscr{L})$, the following are equivalent:*

(1)X is T_1^*-space.
(2)$cl_R^*(\{x\}) = \{x\}$ for each $x \in X$.
(3)$(\{x\})^* \subseteq \{x\}$ for each $x \in X$.

Proof.

(1)$\Rightarrow$ (2): Suppose that $(X,R,\mathscr{L})$ is T_1^*-space and $x \in X$ is an arbitrary point, then for $y \in X - \{x\}, x \neq y$ and $\exists A \subseteq X$ such that $y \in int_R^*(A), x \notin A$. Thus, $y \in ((A^c)^*)^c, x \in A^c$. Thus, $\{x\} - (A^c)^* \subseteq A^c - (A^c)^* \in \mathscr{L}, (A^c)^* = \overline{R}((A^c)^*)$. So,$(\{x\})^* \subseteq (A^c)^*$. Hence, $y \notin \{x\} \cup (\{x\})^* = cl_R^*(\{x\})$ for any arbitrary point $y \in X - \{x\}$. Hence,

$$cl_R^*(\{x\}) = \{x\}.$$

(2)⇒ (3): For each $x \in X$, by (2), $cl_R^*(\{x\}) = \{x\}, \{x\} - \{x\} \in \mathscr{L}$. Thus, $(\{x\})^* \subseteq \{x\}$.
(3)⇒ (1): For each $x \neq y \in X$. By (3) $(\{x\})^* \subseteq \{x\}, (\{y\})^* \subseteq \{y\}$. Then $cl_R^*(\{x\}) = \{x\}, cl_R^*(\{y\}) = \{y\}$, and $int_R^*(\{x\}^c) = \{x\}^c, int_R^*(\{y\}^c) = \{y\}^c$. Then, $(x \in int_R^*(\{y\}^c), y \notin \{y\}^c)$ and $(y \in int_R^*(\{x\}^c), x \notin \{x\}^c)$,i.e., X is T_1^*-space.

Corollary 22*For an ideal approximation space $(X,R,\mathscr{L})$, the following are equivalent:*

(1)X is $T_1^{**}-$space.
(2)$cl_R^{**}(\{x\}) = \{x\}$ for each $\forall x \in X$.
(3)$(\{x\})^{**} \subseteq \{x\}$ for each $x \in X$.

Theorem 21*For an ideal approximation space $(X,R,\mathscr{L})$, the following are equivalent:*

(1)X is T_2^*-space.
(2)$\exists A \subseteq X : x \in int_R^*(A), y \in (cl_R^*(A))^c$ for all $x \neq y \in X$.

Proof.

(1)⇒ (2): If X is T_2^*-space, then $\forall x \neq y \in X$; there exists $x \in int_R^*(A), y \in int_R^*(B)$ and $A \cap B = \phi$. So, $int_R^*(B) \subseteq int_R^*(A^c) = (cl_R^*(A))^c$. Hence, $x \in int_R^*(A), y \in int_R^*(B) \subseteq (cl_R(A))^c$.
(2)⇒ (1): Let $x \neq y \in X$. Then from(2) $x \in int_R^*(A), y \in (cl_R^*(A))^c$ and clearly $int_R^*(cl_R^*(A))^c = (cl_R^*(A))^c$ and $A \cap (cl_R^*(A))^c = \phi$. Hence X is T_2^*-space.

Corollary 23*For an ideal approximation space $(X,R,\mathscr{L})$, the following are equivalent:*

(1)X is $T_2^{**}-$space.
(2)$\exists A \subseteq X : x \in int_R^{**}(A), y \in (cl_R^{**}(A))^c$ for all $x \neq y \in X$.

Definition 22

(1)An ideal approximation space $(X,R,\mathscr{L})$ is R_0^*-space iff $cl_R^*(\{x\}) = cl_R^*(\{y\})$ or $cl_R^*(\{x\}) \cap cl_R^*(\{y\}) = \phi$ $\forall x \neq y \in X$.
(2)An ideal approximation space $(X,R,\mathscr{L})$ is $R_0^{**}-$space iff $cl_R^{**}(\{x\}) = cl_R^{**}(\{y\})$ or $cl_R^{**}(\{x\}) \cap cl_R^{**}(\{y\}) = \phi$ $\forall x \neq y \in X$.

Proposition 23*For an ideal approximation space $(X,R,\mathscr{L})$, the following are equivalent:*

(1)X is R_0^*-space.
(2)If $x \in cl_R^*(\{y\})$ then $y \in cl_R^*(\{x\})$ for all $x \neq y \in X$.

Proof.
(1)⇒ (2): Let x and y be two distinct points in $(X,R,\mathscr{L})$ and $cl_R^*(\{x\}) = cl_R^*(\{y\})$ or $cl_R^*(\{x\}) \cap cl_R^*(\{y\}) = \phi$. In the former case we have $y \in cl_R^*(\{x\})$ and $x \in cl_R^*(\{y\})$. In the latter case we get $\{x\} \cap cl_R^*(\{y\}) = \phi$ and $\{y\} \cap cl_R^*(\{x\}) = \phi$ which mean that $x \notin cl_R^*(\{y\})$ and $y \notin cl_R^*(\{x\}))$. Hence, $x \notin cl_R^*(\{y\})$ and $y \notin cl_R^*(\{x\})$. So, if $x \in cl_R^*(\{y\})$ then $y \in cl_R^*(\{x\})$.

(2)⇒ (1): If $x \in cl_R^*(\{y\})$ then $y \in cl_R^*(\{x\})$ holds, then either

$$(x \in cl_R^*(\{y\}) \text{ and } y \in cl_R^*(\{x\}))$$

or

$$(x \notin cl_R^*(\{y\}) \text{ and } y \notin cl_R^*(\{x\}))$$

are holds. In the former case we have for $x \neq y \in X$, then

$$cl_R^*(\{x\}) = cl_R^*(\{y\}). \tag{2.1}$$

In the latter case we get for $x \neq y \in X$, then

$$cl_R^*(\{x\}) \cap cl_R^*(\{y\}) = \phi. \tag{2.2}$$

From (2.1) and (2.2) the proof is complete.

Corollary 24*For an ideal approximation space $(X,R,\mathscr{L})$, the following are equivalent:*

(1)X is $R_0^{**}-$space.
(2)If $x \in cl_R^{**}(\{y\})$ then $y \in cl_R^{**}(\{x\})$ for all $x \neq y \in X$.

Corollary 25*For an ideal approximation space $(X,R,\mathscr{L})$, the following are holds,*

(1)$T_1^* = R_0^* + T_0^*$.
(2)$T_1^{**} = R_0^{**} + T_0^{**}$.

*Proof.*Immediately derived from Definition 21, Proposition 23.

Remark 22*We introduce the following examples to show that $R_0^* \not\Leftrightarrow T_0^*$, $R_0^* \not\Leftrightarrow R_0^{**}$, and $R_0^{**} \not\Leftrightarrow T_0^{**}$.*

Example 24

(1)In Example 21 (2) X is T_0^*-space. But we have $\{a\}^* = \phi, \{b\}^* = \{a,b\}$. Then, $cl_R^*(\{a\}) = \{a\} \cup (\{a\})^* = \{a\}, cl_R^*(\{b\}) = \{b\} \cup (\{b\})^* = \{a,b\}$. This means that, $a \in cl_R^*(\{b\})$ but $b \notin cl_R^*(\{a\})$. Therefore, X is not R_0^*-space.
(2)Let $X = \{a,b,c\}, R = \{(a,a),(a,b),(b,a),(b,b),(c,c)\}$ and $\mathscr{L} = \{\phi, \{c\}\}$. Then, $< a > R = \{a,b\}, < b > R = \{a,b\}, < c > R = \{c\}$. Then, $cl_R^*(\{a\}) = cl_R^*(\{b\}) = \{a,b\}$ and $cl_R^*(\{c\}) = \{c\}$. Therefore,

(i) For $a \neq b$, $a \in cl_R^*(\{b\})$ and $b \in cl_R^*(\{a\})$.
(ii) For $b \neq c$, $b \notin cl_R^*(\{c\})$ and $c \notin cl_R^*(\{b\})$.
(iii) For $a \neq c$, $a \notin cl_R^*(\{c\})$ and $c \notin cl_R^*(\{a\})$.

Hence, X is R_0^*-space. But X is not T_0^*-space since we can not find a set $A \subseteq X$ such that $a \in int_R^*(A)$ and not containing b or $b \in int_R^*(A)$ and not containing a.
(3)In Example 22 (1) X is T_0^{**}-space. But we have $\{a\}^{**} = \{a,b\}, \{b\}^{**} = \phi$. Then, $cl_R^{**}(\{a\}) = \{a\} \cup (\{a\})^{**} = \{a,b\}, cl_R^{**}(\{b\}) = \{b\} \cup (\{b\})^{**} = \{b\}$. This means that, $b \in cl_R^{**}(\{a\})$ but $a \notin cl_R^{**}(\{b\})$. Therefore, X is not $R_0^{**}-$space.

(4)In Example 22 (1) if $\mathscr{L}=\{\phi,\{c\}\}$. Then, $cl^{**}_R(\{a\})=cl^{**}_R(\{b\})=\{a,b\}$ and $cl^{**}_R(\{c\})=\{c\}$. Therefore,

(i) For $a\neq b$, $a\in cl^{**}_R(\{b\})$ and $b\in cl^{**}_R(\{a\})$.

(ii) For $b\neq c$, $b\notin cl^{**}_R(\{c\})$ and $c\notin cl^{**}_R(\{b\})$.

(iii) For $a\neq c$, $a\notin cl^{**}_R(\{c\})$ and $c\notin cl^{**}_R(\{a\})$.

Hence, X is R^{**}_0-space. But X is not T^{**}_0-space since $a\neq b\in X$ cannot be separated.

(5)In Example 22 (2) if $\mathscr{L}=\{\phi,\{c\}\}$. Then, $cl^{*}_R(\{a\})=cl^{*}_R(\{b\})=\{a,b\}$ and $cl^{*}_R(\{c\})=\{c\}$. Therefore,

(i) For $a\neq b$, $a\in cl^{*}_R(\{b\})$ and $b\in cl^{*}_R(\{a\})$.

(ii) For $b\neq c$, $b\notin cl^{*}_R(\{c\})$ and $c\notin cl^{*}_R(\{b\})$.

(iii) For $a\neq c$, $a\notin cl^{*}_R(\{c\})$ and $c\notin cl^{*}_R(\{a\})$.

Hence, X is R^{*}_0-space. But, $cl^{**}_R(\{a\})=\{a\},cl^{**}_R(\{b\})=\{a,b\}$. Then, $a\in cl^{**}_R(\{b\}),b\notin cl^{**}_R(\{a\})$. Therefore, X is not R^{**}_0-space.

(6)Let $X=\{a,b,c\}$, $R=\{(a,a),(a,b),(a,c),(b,b),(b,c),(c,c)\}$, and $\mathscr{L}=\{\phi,\{a\},\{c\},\{a,c\}\}$. Then, $<a>R=\{a,b,c\},<b>R=\{b,c\},<c>R=\{c\}$. Also, $R<a>=\{a\},R<b>=\{a,b\},R<c>=\{a,b,c\}$. Therefore, $R<a>R=\{a\},R<b>R=\{b\},R<c>R=\{c\}$.Then, $cl^{**}_R(\{a\})=\{a\},cl^{**}_R(\{b\})=\{b\},cl^{**}_R(\{c\})=\{c\}$. Hence, X is R^{**}_0-space. But, $cl^{*}_R(\{a\})=\{a\},cl^{*}_R(\{b\})=\{a,b\}$. Then, $a\in cl^{*}_R(\{b\}),b\notin cl^{*}_R(\{a\})$. This means that, X is not R^{*}_0-space.

Definition 23

(1)Let (X,R_1) and (Y,R_2) are approximation spaces. Then, a function $f:(X,R_1)\longrightarrow(Y,R_2)$ is said to be continuous if $L_{R_1}(f^{-1}(V))\supseteq f^{-1}(L_{R_1}(V))$ for all $V\in Y$. It is equivalent to $U_{R_1}(f^{-1}(V))\subseteq f^{-1}(U_{R_2}(V))$ for all $V\in Y$.

(2)Let $\mathscr{L}$ be an ideal on X. Then, a function $f:(X,R_1,\mathscr{L})\longrightarrow(Y,R_2)$ is said to be $*-$continuous (resp. $**-$continuous) if $int^{*}_{R_1}(f^{-1}(V))\supseteq f^{-1}(L_{R_1}(V))$ (resp. $int^{**}_{R_1}(f^{-1}(V))\supseteq f^{-1}(L_{R_1}(V))$) for all $V\in Y$. It is equivalent to $cl^{*}_{R_1}(f^{-1}(V))\subseteq f^{-1}(U_{R_2}(V)$ (resp. $cl^{**}_{R_1}(f^{-1}(V))\subseteq f^{-1}(U_{R_2}(V))$ for all $V\in Y$.

Remark 23*From Theorem 01 (3) we have the following diagram:*

$$Continuous \Longrightarrow *-continuous \Longrightarrow **-continuous.$$

Next examples show that the Implication in the diagram is not reversible.

Example 25 *Let $X=\{a,b,c\}$, $R_1=\{(a,a),(a,b),(a,c),(b,b),(b,c)\},Y=\{1,2,3\}$ and $R_2=\{(1,1),(1,2),(2,1),(2,2),(3,3)\}$. Then, $<a>R_1=\{a,b,c\},<b>R_1=\{b,c\},<c>R_1=\{b,c\}$. Also, $R_1<a>=\{a\},R_1<b>=\{a,b\},R_1<c>=\phi$. Therefore, $R_1<a>R_1=\{a\},R_1<b>R_1=\{b\},R_1<c>R_1=\phi$. and, $<1>R_2=\{1,2\},<2>R_2=\{1,2\},<3>R_2=\{3\}$. Also, $R_2<1>=\{1,2\},R_2<2>=\{1,2\},R_2<3>=\{3\}$. Therefore, $R_2<1>R_2=\{1,2\},R_2<2>R_2=\{1,2\},R_2<3>R_2=\{3\}$. Let $f:(X,R_1,\mathscr{L})\longrightarrow(Y,R_2)$ where $f(a)=f(b)=1,f(c)=3$. Then,*

(1)Consider $\mathscr{L}=\{\phi,\{b\},\{c\},\{b,c\}\}$. Then, $int^{*}_{R_1}(f^{-1}(\{1\}))=\{a,b\}\supseteq f^{-1}(L_{R_1}(\{1\}))=\phi$, $int^{*}_{R_1}(f^{-1}(\{2\}))=\phi\supseteq f^{-1}(L_{R_1}(\{2\}))=\phi$, $int^{*}_{R_1}(f^{-1}(\{3\}))=\{c\}\supseteq f^{-1}(L_{R_1}(\{3\}))=\{c\}$, $int^{*}_{R_1}(f^{-1}(\{1,2\}))=\{a,b\}\supseteq f^{-1}(L_{R_1}(\{1,2\}))=\{a,b\}$, $int^{*}_{R_1}(f^{-1}(\{1,3\}))=X\supseteq f^{-1}(L_{R_1}(\{1,3\}))=\{c\}$, and $int^{*}_{R_1}(f^{-1}(\{2,3\}))=\{c\}\supseteq f^{-1}(L_{R_1}(\{2,3\}))=\{c\}$. So, X is $*-$continuous. But X is not continuous since $L_{R_1}(f^{-1}(\{3\}))=\phi\not\supseteq f^{-1}(L_{R_1}(\{3\}))=\{c\}$.

(2)Consider $\mathscr{L}=\{\phi,\{a\}\}$. Then, $int^{**}_{R_1}(f^{-1}(\{1\}))=\{a,b\}\supseteq f^{-1}(L_{R_1}(\{1\}))=\phi$, $int^{**}_{R_1}(f^{-1}(\{2\}))=\phi\supseteq f^{-1}(L_{R_1}(\{2\}))=\phi$, $int^{**}_{R_1}(f^{-1}(\{3\}))=\{c\}\supseteq f^{-1}(L_{R_1}(\{3\}))=\{c\}$, $int^{**}_{R_1}(f^{-1}(\{1,2\}))=\{a,b\}\supseteq f^{-1}(L_{R_1}(\{1,2\}))=\{a,b\}$, $int^{**}_{R_1}(f^{-1}(\{1,3\}))=X\supseteq f^{-1}(L_{R_1}(\{1,3\}))=\{c\}$, and $int^{**}_{R_1}(f^{-1}(\{2,3\}))=\{c\}\supseteq f^{-1}(L_{R_1}(\{2,3\}))=\{c\}$. So, X is $**-$continuous. But X is not $*-$continuous since $int^{**}_{R_1}(f^{-1}(\{1,2\}))=\phi\not\supseteq f^{-1}(L_{R_1}(\{1,2\}))=\{a,b\}$.

Theorem 22*Let $f:(X,R_1)\longrightarrow(Y,R_2)$ be an injective continuous function. Then, $(X,R_1,\mathscr{L})$ is T^{*}_i-space if (Y,R_2) is T_i- space, $i=0,1,2$.*

*Proof.*We proof for $i=2$. Since $x\neq y$ in X implies that $f(x)\neq f(y)$ in Y, and from Y is T_2- space, then there exist $V,W\subseteq Y$ with $f(x)\in L_{R_1}(V)$, $f(y)\in L_{R_1}(W)$ and $V\cap W=\phi$, that is $x\in f^{-1}(L_{R_1}(V))$, $y\in f^{-1}(L_{R_1}(W))$ and $f^{-1}(V)\cap f^{-1}(W)=\phi$. Since f is continuous, then $x\in L_{R_1}(f^{-1}(V))$, $y\in L_{R_1}(f^{-1}(W))$, and then $x\in int^{*}_{R_1}(f^{-1}(V)),y\in int^{*}_{R_1}(f^{-1}(W))$. That is, there exists $A=f^{-1}(V),B=f^{-1}(W)$ in X with $x\in intR_1{}^{*}(A),y\in int^{*}_{R_1}(B)$ and $A\cap B=\phi$. Hence, $(X,R_1,\mathscr{L})$ is T^{*}_2-space. For $i=0,1$, are similar.

Corollary 26*Let $f:(X,R_1)\longrightarrow(Y,R_2)$ be an injective continuous function. Then, $(X,R_1,\mathscr{L})$ is T^{**}_i-space if (Y,R_2) is T_i- space, $i=0,1,2$.*

3 Connectedness in ideal approximation spaces

Definition 31 *Let (X,R) be an approximation space. Then,*

(1) *$A,B \subseteq X$ are called separated sets if $U_R(A) \cap B = A \cap U_R(B) = \phi$.*
(2) *$Y \subseteq X$ is called disconnected set if there exists separated sets $A,B \subseteq X$, such that $Y \subseteq A \cup B$. and, Y is called connected if it is not disconnected.*
(3) *(X,R) is called disconnected space if there exists separated sets $A,B \subseteq X$, such that $A \cup B = X$. An approximation space (X,R) is called connected space if it is not disconnected space.*

Definition 32 *Let $(X,R,\mathscr{L})$ be an ideal approximation space. Then,*

(1) *$A,B \subseteq X$ are called $*-$separated (resp. $**-$separated) sets if $cl^*_R(A) \cap B = A \cap cl^*_R(B) = \phi$ (resp. $cl^{**}_R(A) \cap B = A \cap cl^{**}_R(B) = \phi$).*
(2) *$Y \subseteq X$ is called $*-$disconnected (resp. $**-$disconnected) set if there exists $*-$separated (resp. $**-$separated) sets $A,B \subseteq X$, such that $Y \subseteq A \cup B$. and, Y is called $*-$connected (resp. $**-$connected) if it is not $*-$disconnected (resp. $**-$disconnected).*
(3) *$(X,R,\mathscr{L})$ is called $*-$disconnected (resp. $**-$disconnected) space if there exists $*-$separated (resp. $**-$separated) sets $A,B \subseteq X$, such that $A \cup B = X$. An ideal approximation space $(X,R,\mathscr{L})$ is called $*-$connected (resp. $**-$connected) space if it is not $*-$disconnected (resp. $**-$disconnected) space.*

Remark 31 *(1) We have the following diagrams:*

$$separated \Longrightarrow *-separated \Longrightarrow **-separated.$$

and hence,

$$**-connected \Longrightarrow *-connected \Longrightarrow connected.$$

Next examples show that the Implication in the diagrams is not reversible.

(2) *For $\mathscr{L} = \{\phi\}$, observe that $*-$connected and connected are identical.*

Example 31 *Let $X = \{a,b,c,d\}, R = \{(a,a),(a,b),(b,b),(b,c),(c,c),(d,d),(d,b)\}$ Then, $< a > R = \{a,b\}, < b > R = \{b\}, < c > R = \{c\}, < d > R = \{b,d\}$. Also, $R < a >= \{a\}, R < b >= \{b\}, R < c >= \{b,c\}, R < d >= \{d\}$. Therefore, $R < a > R = \{a\}, R < b > R = \{b\}, R < c > R = \{c\}, R < d > R = \{d\}$. Then,*

(1) *Consider $\mathscr{L} = \{\phi,\{b\}\}$ and $A = \{a,c\}, B = \{b,d\}$. Then $U_R(A) = A \cup \{x \in X :< x > R \cap A \neq \phi\} = \{a,c\}$, and $U_R(B) = \{a,b,d\}$. Also, $A^* = \{a,c\}, B^* = \{d\}$. So, $cl^*_R(A) = A \cup A^* = \{a,c\}$ and $cl^*_R(B) = B \cup B^* = \{b,d\}$. Thus, $cl^*_R(A) \cap B = A \cap cl^*_R(B) = \phi$, but, $A \cap U_R(B) = \{a\} \neq \phi$. Hence, A,B are $*-$separated sets but are not separated sets.*
(2) *Consider $\mathscr{L} = \{\phi,\{d\}\}$ and $A = \{b\}, B = \{a,d\}$. Then $A^* = \{a,b,d\}, B^* = \{a\}$. So, $cl^*_R(A) = A \cup A^* = \{a,b,d\}$ and $cl^*_R(B) = B \cup B^* = \{a,d\}$. Also, $A^{**} = \{b\}, B^{**} = \{a\}$. So, $cl^{**}_R(A) = A \cup A^{**} = \{b\}$ and $cl^{**}_R(B) = B \cup B^{**} = \{a,d\}$. Thus, $cl^{**}_R(A) \cap B = A \cap cl^{**}_R(B) = \phi$, but $cl^*_R(A) \cap B = \{a\} \neq \phi$, Hence, A,B are $**-$separated sets but are not $*-$separated sets.*

Example 32 *Let $X = \{a,b,c\}, R = \{(a,a),(a,b),(a,c),(b,b),(b,c)\}$. Then, $< a > R = \{a,b,c\}, < b > R = \{b,c\}, < c > R = \{b,c\}$. Also, $R < a >= \{a\}, R < b >= \{a,b\}, R < c >= \phi$. Therefore, $R < a > R = \{a\}, R < b > R = \{b\}, R < c > R = \phi$. Then,*

(1) *Consider $\mathscr{L} = \{\phi,\{b\},\{c\},\{b,c\}\}$. Here the space X is connected space because, $U_R(\{b\}) = U_R(\{c\}) = U_R(\{b,c\}) = U_R(\{a,b\}) = U_R(\{a,c\}) = X$ and $U_R(\{a\}) = \{a\}$. But is not $*-$connected space, since $X = \{a\} \cup \{b,c\}$, $cl^*_R(\{a\}) \cap \{b,c\} = \{a\} \cap cl^*_R(\{b,c\}) = \phi$.*
(2) *Consider $\mathscr{L} = \{\phi,\{a\}\}$. Here the space X is $*-$connected space because, $cl^*_R(\{b\}) = cl^*_R(\{c\}) = cl^*_R(\{b,c\}) = cl^*_R(\{a,b\}) = cl^*_R(\{a,c\}) = X$ and $cl^*_R(\{a\}) = \{a\}$. But is not $**-$connected space, since $X = \{a\} \cup \{b,c\}$, $cl^{**}_R(\{a\}) \cap \{b,c\} = \{a\} \cap cl^{**}_R(\{b,c\}) = \phi$.*

Proposition 31 *Let $(X,R,\mathscr{L})$ be an ideal approximation space. Then, the following are equivalent.*

(1) *X is $*-$connected,*
(2) *For each $A,B \subseteq X$ with $A \cap B = \phi, int^*_R(A) = A, int^*_R(B) = B$ and $A \cup B = X$, then $A = \phi$ or $B = \phi$,*
(3) *For each $A,B \subseteq X$ with $A \cap B = \phi, cl^*_R(A) = A, cl^*_R(B) = B$ and $A \cup B = X$, then $A = \phi$ or $B = \phi$.*

Proof. (1)$\Rightarrow$ (2): Let $A,B \subseteq X$ with $int^*_R(A) = A, int^*_R(B) = B$ such that $A \cap B = \phi$ and $A \cup B = X$. Then,

$$cl^*_R(A) \subseteq cl^*_R(B^c) = (int^*_R(B))^c = B^c,$$

$$cl^*_R(B) \subseteq cl^*_R(A^c) = (int^*_R(A))^c = A^c.$$

Hence, $cl^*_R(A) \cap B = A \cap cl^*_R(B) = \phi$. That is, A,B are $*-$separated sets so that $A \cup B = X$. But, $(X,R,\mathscr{L})$ is $*-$connected implies that $A = \phi$ or $B = \phi$.

(2)$\Rightarrow$ (3)Let $A,B \subseteq X$ with $A \cap B = \phi, cl^*_R(A) = A, cl^*_R(B) = B$ and $A \cup B = X$. Then, $cl^*_R(A) \cap B = A \cap cl^*_R(B) = \phi$, thus $int^*_R(A) \cap B = A \cap int^*_R(B) = \phi$. So, $int^*_R(A) = A$ and $int^*_R(B) = B$. Hence, $A = \phi$ or $B = \phi$, from (2).

(3)$\Rightarrow$ (1) Directly from (2), there are no two proper $*-$separated sets $A,B \subseteq X$ such that $A \cup B = X$. Therefore, $(X,R,\mathscr{L})$ is $*-$connected.

Corollary 31 *Let (X,R) be an approximation space. Then, the following are equivalent.*

(1)X is connected,
(2)For each $A,B\subseteq X$ *with* $A\cap B=\phi, L_R(A)=A, L_R(B)=B$ *and* $A\cup B=X$*, them* $A=\phi$ *or* $B=\phi$*,*
(3)For each $A,B\subseteq X$ *with* $A\cap B=\phi, U_R(A)=A, U_R(B)=B$ *and* $A\cup B=X$*, then* $A=\phi$ *or* $B=\phi$*.*

Corollary 32*Let* $(X,R,\mathscr{L})$ *be an ideal approximation space. Then, the following are equivalent.*

(1)X is $**-$*ideal connected,*
(2)For each $A,B\subseteq X$ *with* $A\cap B=\phi, int^{**}_R(A)=A, int^{**}_R(B)=B$ *and* $A\cup B=X$*, then* $A=\phi$ *or* $B=\phi$*,*
(3)For each $A,B\subseteq X$ *with* $A\cap B=\phi, cl^{**}_R(A)=A, cl^{**}_R(B)=B$ *and* $A\cup B=X$*, then* $A=\phi$ *or* $B=\phi$*.*

Theorem 31*Let* $(X,R,\mathscr{L})$ *be an ideal approximation space and* $M\subseteq X$ *is* $*-$*connected. If* $A,B\subseteq X$ *are* $*-$*separated sets with* $M\subseteq A\cup B$*, then either* $M\subseteq A$ *or* $M\subseteq B$*.*

*Proof.*Let A,B are $*-$separated sets with $M\subseteq A\cup B$. Thus, $cl^*_R(A)\cap B=A\cap cl^*_R(B)=\phi$, and $M=(M\cap A)\cup(M\cap B)$. Since, $cl^*_R(M\cap A)\cap(M\cap B)\subseteq cl^*_R(M)\cap cl^*_R(A)\cap(M\cap B)=cl^*_R(M)\cap M\cap cl^*_R(A)\cap B=M\cap\phi=\phi$.

By similar way,
$cl^*_R(M\cap B)\cap(M\cap A)\subseteq cl^*_R(M)\cap cl^*_R(B)\cap(M\cap A)=cl^*_R(M)\cap M\cap cl^*_R(B)\cap A=M\cap\phi=\phi$.

Then, $(M\cap A)$ and $(M\cap B)$ are $*-$separated sets with $M=(M\cap A)\cup(M\cap B)$. But M is $*-$connected implies that $M\subseteq A$ or $M\subseteq B$.

Corollary 33*Let* (X,R) *be an ideal approximation space and* $M\subseteq X$ *is connected. If* $A,B\subseteq X$ *are separated sets with* $M\subseteq A\cup B$*, then either* $M\subseteq A$ *or* $M\subseteq B$*.*

Corollary 34*Let* $(X,R,\mathscr{L})$ *be an ideal approximation space and* $M\subseteq X$ *is* $**-$*ideal connected. If* $A,B\subseteq X$ *are* $**-$*ideal separated sets with* $M\subseteq A\cup B$*, then either* $M\subseteq A$ *or* $M\subseteq B$*.*

Theorem 32*Let* $f:(X,R_1,\mathscr{L})\longrightarrow(Y,R_2)$ *be an* $*-$*continuous function. Then,* $f(A)\subseteq Y$ *is connected set if* A *is* $*-$*connected in* X*.*

*Proof.*Let $f(A)$ be disconnected. Then, there exists two separated sets $U,V\subseteq Y$ with $f(A)\subseteq U\cup V$. That is, $U_{R_2}(U)\cap V=U\cap U_{R_2}(V)=\phi$. Then, $A\subseteq f^{-1}(U)\cup f^{-1}(V)$, and since f is $*-$continuous, we get that:
$cl^*_{R_1}(f^{-1}(U))\cap f^{-1}(V)\subseteq f^{-1}(U_{R_2}(U))\cap f^{-1}(V)=f^{-1}(U_{R_2}(U)\cap V)=f^{-1}(\phi)=\phi$
and in similar way, we have
$cl^*_{R_1}(f^{-1}(V))\cap f^{-1}(U)\subseteq f^{-1}(U_{R_2}(V))\cap f^{-1}(U)=f^{-1}(U_{R_2}(V)\cap U)=f^{-1}(\phi)=\phi$.

Hence, $f^{-1}(U)$ and $f^{-1}(V)$ are $*-$separated sets in X so that $A\subseteq f^{-1}(U)\cup f^{-1}(V)$. so A is $*-$disconnected, which contradicts that A is $*-$connected, this is because of the incorrect assumption that $f(A)$ is disconnected and so $f(A)$ is connected set.

Corollary 35*Let* $f:(X,R_1,\mathscr{L})\longrightarrow(Y,R_2)$ *be an* $**-$*continuous function. Then,* $f(A)\subseteq Y$ *is connected set if* A *is* $**-$*connected in* X*.*

Here, we modify Definition of $<x>R$ and $R<x>$ in defining a new type of roughness of ordinary sets. By the previous definitions of $xR, Rx\in P(X)$, we can define the maximal neighborhoods of any $x\in X$.

Define for any $x\in X$, the sets $<x>\breve{R}, \breve{R}<x>\in X$ as follow:

$$<x>\breve{R}=\begin{cases}\cup_{x\in yR}(yR) & \text{if } \exists y: x\in yR,\\ \phi & o.w.\end{cases}$$

and

$$\breve{R}<x>=\begin{cases}\cup_{x\in yR}(Ry) & \text{if } \exists y: x\in Ry,\\ \phi & o.w.\end{cases}\tag{3.1}$$

Equation 3.1 defines the maximal right and the maximal left neighborhoods of $x\in X$ as used in [22, 23], and more over the authors in [24, 25] introduced new types of rough sets of an ideal approximation space using the maximal right and the maximal left neighborhoods accompanied with an ideal on X. Using Equation 3.1, we can define a new type of rough sets of an ideal approximation space $(X,R,\mathscr{L})$ as follow:

Definition 33*Let* $(X,R,\mathscr{L})$ *be any ideal approximation space. Then, define the local closed sets* $\Phi(A),\Psi(A)\in P(X)$ *of a set* $A\subseteq X$ *as follow:*

$$\Phi(A)=\bigcap\{G\subseteq X: A-G\in\mathscr{L}, \overline{Uppr}(A)=G\}.\tag{3.2}$$

$$\Psi(A)=\bigcap\{G\subseteq X: A-G\in\mathscr{L}, \overline{\overline{Uppr}}=G\}.\tag{3.3}$$

where

$$\overline{Uppr}(A)=A\cup\{x\in X\ :<x>\breve{R}\cap A\notin\mathscr{L}\},$$

$$\overline{\overline{Uppr}}(A)=A\cup\{x\in X:\breve{R}<x>\breve{R}\cap A\notin\mathscr{L}\}.$$

where $\breve{R}<x>\breve{R}$ *is defined as:*

$$\breve{R}<x>\breve{R}=\breve{R}<x>\cap<x>\breve{R}.$$

The roughness of the set $A\subseteq X$ *is defined by:*

$$Cl^*_R(A)=A\cup\Phi(A),\ Int^*_R(A)=A\cap(\Phi(A^c))^c$$
$$and\ \ Cl^{**}_R(A)=A\cup\Psi(A),\ Int^{**}_R(A)=A\cap(\Psi(A^c))^c.$$

$Int^*_R(A), Int^{**}_R(A)$ *are the lower sets of* A *and* $Cl^*_R(A), Cl^{**}_R(A)$ *are the upper sets of* A

Definition 33 is a generalization of rough sets in a similar way to that defined in Definition 12. Also, this definition introduces the next topologies:

$$\top^* = \{A \subseteq X : Int^*_R(A) = A\} = \{A \subseteq X : Cl^*_R(A^c) = A^c\} \tag{3.4}$$

$$\top^{**} = \{A \subseteq X : Int^{**}(A) = A\} = \{A \subseteq X : Cl^{**}_R(A^c) = A^c\} \tag{3.5}$$

on the ideal approximation space $(X, R, \mathscr{L})$. All topological properties could be studied as in Section (3) and Section (4). In fact, Separation axioms defined on an ideal approximation space $(X, R, \mathscr{L})$ based on the maximal neighborhoods imply the corresponding separation axioms defined in Section (3), and the converse is not true in general. Also, connectedness based on the maximal neighborhoods imply the corresponding connectedness defined in Section (4), and the converse is not true in general.

Example 33

(1)Let $X = \{a,b,c\}, R = \{(a,a),(a,b),(b,b),(c,c)\}$ and $\mathscr{L} = \{\phi, \{c\}\}$. Then, $< a > R = \{a,b\}, < b > R = \{b\}, < c > R = \{c\}$. Also, $< a > \breve{R} = \{a,b\}, < b > \breve{R} = \{a,b\}, < c > \breve{R} = \{c\}$. Thus, $int^*_R(\{b\}) = \{b\} \cap ((((\{b\})^c))^*)^c = \{b\} \cap \{b,c\} = \{b\}, int^*_R(\{c\}) = \{c\} \cap ((((\{b\})^c))^*)^c = \{c\} \cap \{c\} = \{c\}$. Then, for $a \neq b, b \neq c$ there exists $\{b\} \subseteq X$ such that $b \in int^*_R(\{b\})$ and $a,c \notin \{b\}$. For $a \neq c$ there exists $\{c\} \subseteq X$ such that $c \in int^*_R(\{c\})$ and $a \notin \{c\}$. Hence, X is T^*_0-space but not T_0space in sense of Definition 33 because we can not find a set $A \subseteq X$ such that $a \in Int^*_R(A)$ and not containing b or $b \in Int^*_R(A)$ and not containing a.

(2)Let $X = \{a,b,c\}$, $R = \{(a,a),(a,b),(b,b),(b,c),(c,c),(c,a)\}$. Then, $< a > R = \{a\}, < b > R = \{b\}, < c > R = \{c\}$. Also, $< a > \breve{R} =< b > \breve{R} =< c > \breve{R} = \{a,b,c\}$. Consider an ideal $\mathscr{L}$ on X defined by $\mathscr{L} = \{\phi,\{b\},\{c\},\{b,c\}\}$. $\mathscr{L} = \{\phi,\{b\},\{c\},\{b,c\}\}$, Then, there exist $A = \{a\}, B = \{b\}, C = \{c\}$ so that $int^*_R(\{a\}) = \{a\}, int^*_R(\{b\}) = \{b\}, int^*_R(\{c\}) = \{c\}$. Then for $a \neq b$ we have $A,B \subseteq X$ such that $a \in int^*_R(A) = \{a\}, b \notin A$ and $b \in Int^*_R(B) = \{b\}, a \notin B$ and $A \cap B = \phi$. Similarly for $a \neq c, b \neq c$. Hence, X is T^*_1 and T^*_2-space. But X is neither T_1-space nor T^-_2space in sense of Definition 33 because we can not find $A,B \subseteq X$ such that $a \in int^*_R(A), b \notin A$ and $b \in int^*_R(B), a \notin B$.

Example 34*In Example 32 $(X,R,\mathscr{L})$ is $*-$disconnected space. But, $< a > \breve{R} =< b > \breve{R} =< c > \breve{R} = \{a,b,c\}$. Then, $Cl^*_R(\{a\}) = Cl^*_R(\{a,b\}) = Cl^*_R(\{a,c\}) = X$. Therefore, we can not find two separated sets $A,B \subseteq X$ to make $(X,R,\mathscr{L})$ a connected space in sense of Definition 33.*

4 Conclusions

Here, we introduced separation axioms, connectedness, and continuity in ideal approximation spaces generated by using local functions based on minimal neighborhoods. Moreover, we modified definitions to get a new pattern of approximation spaces based on maximal neighborhoods. In addition, some examples are given to explain the relationship between the two types. In a proposed future work, a new generalization of rough fuzzy sets based on a fuzzy ideal $\mathscr{L}$ will be introduced on a fuzzy approximation space (X,R) (See, [26]).

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