Sohag Journal of Science
An International Journal

Strong coupled fixed point results in fuzzy cone metric spaces

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Received: 19 Mar. 2022, Revised: 5 Apr. 2022, Accepted: 18 Apr. 2022
Published online: 1 Sep. 2022.

Abstract: In this paper, strong coupled fixed point theorems are obtained for coupled Kannan-type contraction mappings in the setting of fuzzy cone metric spaces. Moreover, to support our results, non-trivial examples are given. Our results generalize and extend a lot of papers in the literatures.

Keywords: Strong coupled fixed point; fuzzy cone metric space; contraction conditions; cyclic coupled type fuzzy cone contraction mapping.

1 Introduction

Fixed point theory is quite useful in the existence theory of differential, integral, partial differential and functional equations. This is a basic mathematical tool used in obtaining the existence of solutions of problems in mathematical economics theory, nonlinear analysis, topology, control theory, dynamical system, functional analysis, differential equations, global analysis and game theory, etc. Moreover, it is a very important tool used to find analytical and numerical solutions in nonlinear problems which shown in mathematical methods, game theory, biology, engineering and physics, see [1,2,3,4,5,6].

In 2003, Kirk et al. [7] introduced fixed point results under cyclic contractive conditions. Lakshmikantham and Ćirić [8] established the concept of coupled fixed points (CFPs) in the context of partially ordered metric spaces. Also, they discussed the existence and uniqueness of the solution of periodic boundary value problems. For more details, see [9,10,11,12,13,14,15,16].

Huang and Zhang [17] presented the idea of cone metric spaces by replacing real numbers with an ordered Banach space and showed some fixed point results in such spaces. Moreover, the theory of fuzzy sets improved by Zadeh [18]. In particular, Kramosil and Michalek in [19] presented a fuzzy metric space. Many authors have investigated fixed point theorems and common fixed point theorems in cone metric spaces, see [20,21,22,23,24,25,26].

2 Preliminaries

Definition 1.*[20] A subset $\Upsilon \in \digamma$ describes a cone if:*

(1)$\Upsilon \neq \emptyset$, closed and $\Upsilon \neq \{\vartheta\}$;
(2)$\lambda_1, \beta_1 \in (0,\infty)$ and $\theta, \rho \in \Upsilon$, then $\lambda_1\theta + \beta_1\rho \in \Upsilon$;
(3)both $\theta - \theta \in \Upsilon$, then $\theta = \vartheta$.

A partial ordering on a given cone $\Upsilon \subset \digamma$ is given by $\theta \leq \rho \Longleftrightarrow \rho - \theta \in \Upsilon$. $\theta < \rho$ symbolize $\theta \leq \rho$ and $\theta \neq \rho$, while $\theta \ll \rho$ symbolize $\rho - \theta \in \Upsilon^0$, where Υ^0 stands for the interior of Υ, it should be noted that all cones have non-empty interior.

Here, $\digamma$ is the real Banach space and ϑ represents a zero element in $\digamma$.

Definition 2.*[25] A trio $(\Omega, \Theta_\varpi, *)$ is called a fuzzy cone metric space (FCMS) if a cone $\Upsilon \in \digamma$, Ω is an arbitrary set, $*$ is a continuous $\upsilon-$norm, and Θ_ϖ is a fuzzy set on $\Omega^2 \times \Upsilon^0$ so that the assertions below hold:*

(1)$\Theta_\varpi(\theta,\rho,\upsilon) > \vartheta$ and $\Theta_\varpi(\theta,\rho,\upsilon) = 1$ iff $\theta = \rho$;
(2)$\Theta_\varpi(\theta,\rho,\upsilon) = \Theta_\varpi(\rho,\theta,\upsilon)$;
(3)$\Theta_\varpi(\theta,\rho,\upsilon) * \Theta_\varpi(\rho,\delta,\mu) \leq \Theta_\varpi(\theta,\delta,\upsilon+\mu)$;
(4)$\Theta_\varpi(\theta,\rho,\cdot) : \Upsilon^0 \to [0,1]$ is continuous,

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for all $\theta,\rho,\delta\in\Omega$ *and* $\upsilon,\mu\in\Upsilon^0$.

Definition 3.[17] *Assume that* $(\Omega,\Theta_\varpi,*)$ *is a FCMS,* $\theta\in\Omega$ *and* (θ_i) *is a sequence in* Ω. *Then*

(a) (θ_i) *is called converge to* θ *if, for* $\upsilon\gg\vartheta$ *and* $0<u<1$, $\exists i_1\in\mathbb{N}$ *so that* $\Theta_\varpi(\theta_i,\theta,\upsilon)>1-u$, $\forall i>i_1$, *and we can write* $\lim_{i\to\infty}\theta_i=\theta$ *or* $\theta_i\to\theta$ *as* $i\to\infty$;

(b) (θ_i) *is called a Cauchy sequence if, for* $\upsilon\gg\vartheta$ *and* $0<u<1$, $\exists i_1\in\mathbb{N}$ *so that*

$$\Theta_\varpi(\theta_k,\theta_i,\upsilon)>1-u,\ \forall k,i>i_1;$$

(c) *if every Cauchy sequence is convergent in* Ω, *then we say a trio* $(\Omega,\Theta_\varpi,*)$ *is complete;*

(d) (θ_i) *is known as a fuzzy cone contraction (FCC) if* $\exists\beta\in(0,1)$ *so that*

$$\left(\frac{1}{\Theta_\varpi(\theta_i,\theta_{i+1},\nu)}-1\right)\le\beta\left(\frac{1}{\Theta_\varpi(\theta_{i-1},\theta_i,\upsilon)}-1\right),\ \forall\upsilon\gg\vartheta,\ i\ge1.$$

Definition 4.[26] *Assume that* $(\Omega,\Theta_\varpi,*)$ *is an FCMS, the fuzzy cone metric* Θ_ϖ *is triangular for all* $\theta,\rho,\delta\in\Omega$, $\upsilon\gg\vartheta$ *if*

$$\left(\frac{1}{\Theta_\varpi(\theta,\delta,\upsilon)}-1\right)\le\left(\frac{1}{\Theta_\varpi(\theta,\rho,\upsilon)}-1\right)+\left(\frac{1}{\Theta_\varpi(\rho,\delta,\upsilon)}-1\right).$$

Lemma 1.[22] *Assume that* $(\Omega,\Theta_\varpi,*)$ *is an FCMS,* $\theta\in D$ *and* (θ_i) *is a sequence in* Ω, *then*

$$\theta_i\to\theta\Leftrightarrow\lim_{i\to\infty}\Theta_\varpi(\theta_i,\theta,\upsilon)=1,\ \textit{for}\ \upsilon\gg\vartheta.$$

Definition 5.*Suppose that* D *and* G *are two non-empty closed subsets of a given set* Ω. *A mapping* $\Xi:\Omega^2\to\Omega$, *so that* $\Xi(\theta,\rho)\in D$ *if* $\theta\in G$, $\rho\in D$ *and* $\Xi(\theta,\rho)\in G$ *if* $\theta\in D$, $\rho\in G$ *is said to be a cyclic map w.r.t.* D *and* G.

Definition 6.*Suppose that* Ω *is a non-empty set. A pair* $(\theta,\rho)\in\Omega^2$ *is called a CFP of the mapping* $\Xi:\Omega^2\to\Omega$ *if* $\Xi(\theta,\rho)=\theta$, $\Xi(\rho,\delta)=\rho$ *and it is said to be a strong CFP if* $\theta=\rho$, *that is* $\Xi(\theta,\theta)=\theta$.

Definition 7.*Suppose that* D *and* G *are two non-empty closed subsets of a FCMS* $(\Omega,\Theta_\varpi,*)$, *where* Θ_ϖ *is triangular and the mapping* $\Xi:\Omega^2\to\Omega$ *is known as a cyclic coupled Kannan-type FCC w.r.t.* D *and* G. *Let* Ξ *verifies*

$$\left(\frac{1}{\Theta_\varpi(\Xi(\theta,\rho),\Xi(q,s),\upsilon)}-1\right)\le\sigma\left(\begin{array}{c}\frac{1}{\Theta_\varpi(\theta,\Xi(\theta,\rho),\upsilon)}-1\\+\frac{1}{\Theta_\varpi(q,\Xi(q,s),\upsilon)}-1\end{array}\right),$$

where $\sigma\in(0,\frac{1}{2})$, and $\theta,s\in D$ and $\rho,q\in G$, for $\upsilon\gg\vartheta$.

Theorem 1.*Suppose that* D *and* G *are two non-empty closed subsets of a complete fuzzy cone metric space (CFCMS)* $(\Omega,\Theta_\varpi,*)$, *where* Θ_ϖ *is triangular and the mapping* $\Xi:\Omega^2\to\Omega$ *is a generalized cyclic coupled Kannan-type contraction w.r.t.* D *and* G *and* $D\cap G=\emptyset$. *Then,* Ξ *has a strong CFP in* $D\cap G$.

3 Main Results

Assume that $D$and $G$are two non-empty closed subsets of a CFCMS $(\Omega,\Theta_\varpi,*)$,where $\Theta_\varpi$is triangular and the mapping $\Xi:\Omega^2\to\Omega$is a generalized cyclic coupled fuzzy cone contractive-type condition w.r.t. $D$and G.Assume that Ξverifies

$$\left(\frac{1}{\Theta_\varpi(\Xi(\theta,\rho),\Xi(q,s),\upsilon)}-1\right)\le\gamma\left(\begin{array}{c}\frac{1}{\Theta_\varpi(\theta,\Xi(\theta,\rho),\upsilon)}-1\\+\frac{1}{\Theta_\varpi(q,\Xi(q,s),\upsilon)}-1\end{array}\right)+\delta\left(\begin{array}{c}\frac{1}{\Theta_\varpi(\theta,\Xi(q,s),\upsilon)}-1\\+\frac{1}{\Theta_\varpi(q,\Xi(\theta,\rho),\upsilon)}-1\end{array}\right),\quad(1)$$

where $\theta,s\in D$ and $\rho,q\in G$, for $\upsilon\gg\vartheta$, and $\gamma,\delta\in[0,\infty)$. Our results generalize and improve Theorem 1.

Our first result in this part is as follows:

Theorem 2.*Suppose that* D *and* G *are two non-empty closed subsets of a CFCMS* $(\Omega,\Theta_\varpi,*)$, *where* Θ_ϖ *is triangular and the mapping* $\Xi:\Omega^2\to\Omega$ *is a cyclic w.r.t.* D *and* G. *Suppose that* Ξ *satisfies* (1) *with* $(\gamma+\delta)<\frac{1}{2}$. *Then* $D\cap G=\emptyset$ *and* Ξ *has a strong CFP in* $D\cap G$.

*Proof.*Define $\theta_0\in D$ and $\rho_0\in G$. Let (θ_i) and (ρ_i) be two sequences given as follows:

$$\theta_{i+1}=\Xi(\rho_i,\theta_i)\text{ and }\rho_{i+1}=\Xi(\theta_i,\rho_i),\quad(2)$$

for all $i\ge\vartheta$. Then $(\theta_i)\subset D$ and $(\rho_i)\subset G$ since Ξ is a cyclic mapping w.r.t. D and G. Let us denote

$$\xi=\frac{\gamma+\delta}{1-(\gamma+\delta)}.$$

Then $\xi\in(0,1)$ for $(\gamma+\delta)<\frac{1}{2}$. We prove that, for $\upsilon\gg\vartheta$ and $i\ge\vartheta$,

$$\left(\frac{1}{\Theta_\varpi(\rho_i,\theta_{i+1},\upsilon)}-1\right)+\left(\frac{1}{\Theta_\varpi(\theta_i,\rho_{i+1},\upsilon)}-1\right)\le\xi^i\left(\frac{1}{\Theta_\varpi(\rho_0,\theta_1,\upsilon)}-1+\frac{1}{\Theta_\varpi(\theta_0,\rho_1,\upsilon)}-1\right).\quad(3)$$

Clearly, (3) holds for $i=\vartheta$. Suppose that (3) holds for $i=k$, $\upsilon \gg \vartheta$, then from (1), we have

$$\begin{aligned}
&\left(\frac{1}{\Theta_{\varpi}(\rho_{k+1},\theta_{k+2},\upsilon)}-1\right)\\
&=\left(\frac{1}{\Theta_{\varpi}(\Xi(\theta_k,\rho_k),\Xi(\rho_{k+1},\theta_{k+1}),\upsilon)}-1\right)\\
&\le \gamma\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\theta_k,\Xi(\theta_k,\rho_k),\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\rho_{k+1},\Xi(\rho_{k+1},\theta_{k+1}),\upsilon)}-1\end{pmatrix}\\
&\quad+\delta\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\rho_{k+1},\Xi(\theta_k,\rho_k),\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\theta_k,\Xi(\rho_{k+1},\theta_{k+1}),\upsilon)}-1\end{pmatrix},\\
&\le \gamma\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\theta_k,\rho_{k+1},\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\rho_{k+1},\theta_{k+2},\upsilon)}-1\end{pmatrix}\\
&\quad+\delta\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\rho_{k+1},\rho_{k+1},\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\theta_k,\theta_{k+2},\upsilon)}-1\end{pmatrix}\\
&\le \gamma\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\theta_k,\rho_{k+1},\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\rho_{k+1},\theta_{k+2},\upsilon)}-1\end{pmatrix}\\
&\quad+\delta\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\theta_k,\rho_{k+1},\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\rho_{k+1},\theta_{k+2},\upsilon)}-1\end{pmatrix},
\end{aligned}$$

which implies that

$$\left(\frac{1}{\Theta_{\varpi}(\rho_{k+1},\theta_{k+2},\nu)}-1\right)\le \xi\left(\frac{1}{\Theta_{\varpi}(\theta_k,\rho_{k+1},\nu)}-1\right),$$

for $\upsilon \gg \vartheta$. Similarly, based on *(1)*, one can write

$$\left(\frac{1}{\Theta_{\varpi}(\theta_{k+1},\rho_{k+2},\upsilon)}-1\right)\le \xi\left(\frac{1}{\Theta_{\varpi}(\rho_k,\theta_{k+1},\upsilon)}-1\right),$$

for $\upsilon \gg \vartheta$. Thus, by mathematical induction inequality (3) is satisfied. Based on (3) with $i=k$, we get

$$\begin{aligned}
&\left(\frac{1}{\Theta_{\varpi}(\rho_{k+1},\theta_{k+2},\upsilon)}-1\right)+\left(\frac{1}{\Theta_{\varpi}(\theta_{k+1},\rho_{k+2},\upsilon)}-1\right)\\
&\le \xi\begin{pmatrix}\left(\frac{1}{\Theta_{\varpi}(\rho_k,\theta_{k+1},\upsilon)}-1\right)\\+\left(\frac{1}{\Theta_{\varpi}(\theta_k,\rho_{k+1},\upsilon)}-1\right)\end{pmatrix}\\
&\qquad\le\cdots\le \xi^{k+1}\begin{pmatrix}\left(\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1\right)\\+\left(\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\right)\end{pmatrix}.
\end{aligned}$$

Hence (3) is fulfilled for $i=k+1$. Therefore (3) holds, for all $i\ge\vartheta$. Also, by (1) for $i\ge\vartheta$, we can write

$$\begin{aligned}
&\left(\frac{1}{\Theta_{\varpi}(\rho_i,\rho_{i+1},\upsilon)}-1\right)+\left(\frac{1}{\Theta_{\varpi}(\theta_i,\theta_{i+1},\upsilon)}-1\right)\\
&\le\left(\frac{1}{\Theta_{\varpi}(\rho_i,\theta_{i+1},\upsilon)}-1\right)+\left(\frac{1}{\Theta_{\varpi}(\theta_{i+1},\rho_{i+1},\upsilon)}-1\right)\\
&\quad+\left(\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1\right)+\left(\frac{1}{\Theta_{\varpi}(\rho_{i+1},\theta_{i+1},\upsilon)}-1\right)\\
&=\left(\frac{1}{\Theta_{\varpi}(\rho_i,\theta_{i+1},\upsilon)}-1\right)+\left(\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1\right)\\
&\quad+2\left(\frac{1}{\Theta_{\varpi}(\Xi(\theta_i,\rho_i),\Xi(\rho_i,\theta_i),\upsilon)}-1\right),\\
&\le\left(\frac{1}{\Theta_{\varpi}(\rho_i,\theta_{i+1},\upsilon)}-1\right)+\left(\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1\right)\\
&\quad+2\gamma\begin{pmatrix}\left(\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1\right)\\+\left(\frac{1}{\Theta_{\varpi}(\rho_i,\theta_{i+1},\upsilon)}-1\right)\end{pmatrix}\\
&\quad+2\delta\begin{pmatrix}\left(\frac{1}{\Theta_{\varpi}(\rho_i,\rho_{i+1},\upsilon)}-1\right)\\+\left(\frac{1}{\Theta_{\varpi}(\theta_i,\theta_{i+1},\upsilon)}-1\right)\end{pmatrix},
\end{aligned}$$

then, we get

$$\begin{aligned}
&\left(\frac{1}{\Theta_{\varpi}(\theta_i,\theta_{i+1},\upsilon)}-1\right)+\left(\frac{1}{\Theta_{\varpi}(\rho_i,\rho_{i+1},\upsilon)}-1\right)\\
&\le(1+2\gamma)\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\rho_i,\theta_{i+1},\upsilon)}-1\end{pmatrix}\\
&\quad+2\delta\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\theta_i,\theta_{i+1},\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\rho_i,\rho_{i+1},\upsilon)}-1\end{pmatrix}.
\end{aligned}$$

This together with *(3) satisfies that*

$$\begin{aligned}
&\left(\frac{1}{\Theta_{\varpi}(\rho_i,\rho_{i+1},\upsilon)}-1\right)+\left(\frac{1}{\Theta_{\varpi}(\theta_i,\theta_{i+1},\upsilon)}-1\right)\\
&\le\frac{(1+2\gamma)}{(1-2\delta)}\xi^i\begin{pmatrix}\left(\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1\right)\\+\left(\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\right)\end{pmatrix},
\end{aligned}\tag{4}$$

for $\upsilon \gg \vartheta$. Thus, for $i,j\ge\vartheta$, without loss of generally, let $i\le j$,

$$\begin{aligned}
&\left(\frac{1}{\Theta_{\varpi}(\rho_i,\rho_j,\upsilon)}-1\right)\\
&\le\sum_{k=i}^{j-1}\left(\frac{1}{\Theta_{\varpi}(\rho_i,\rho_{i+1},\upsilon)}-1\right)\\
&\le\sum_{k=i}^{j-1}\frac{(1+2\gamma)}{(1-2\delta)}\xi^i\begin{pmatrix}\left(\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1\right)\\+\left(\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\right)\end{pmatrix}\\
&=\frac{(1+2\gamma)}{(1-2\delta)(1-\xi)}\xi^i\begin{pmatrix}\left(\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1\right)\\+\left(\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\right)\end{pmatrix}\\
&\to\vartheta,\ \text{as } i\to\infty.
\end{aligned}$$

This proves that (ρ_i) is a Cauchy sequence and convergent in Ω. Because D and G are non-empty closed subsets of Ω, we can write

$$\rho_i\to\rho\in G,\ \text{as } i\to\infty. \tag{5}$$

Analogously,

$$\theta_i\to\theta\in D,\ \text{as } i\to\infty, \tag{6}$$

Then, from (5) and (6), one sees that

$\lim_{i\to\infty} \Theta_{\varpi}(\rho_i,\theta_i,\upsilon) = \Theta_{\varpi}(\rho,\theta,\upsilon)$, and

$\lim_{i\to\infty} \Theta_{\varpi}(\theta_i,\rho_i,\upsilon) = \Theta_{\varpi}(\theta,\rho,\upsilon)$.

Since Θ_{ϖ} is a traingular, by (3) and (4), we get

$$\begin{aligned}&\left(\frac{1}{\Theta_{\varpi}(\rho_i,\theta_i,v)}-1\right)\\ &\le \left(\frac{1}{\Theta_{\varpi}(\rho_i,\rho_{i+1},\upsilon)}-1\right)+\left(\frac{1}{\Theta_{\varpi}(\rho_{i+1},\theta_i,\upsilon)}-1\right)\\ &\le \left(\frac{(1+2\gamma)}{(1-2\delta)}+1\right)\xi^i\begin{pmatrix}\left(\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1\right)\\ +\left(\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\right)\end{pmatrix}\\ &\to \vartheta, \text{ as } i\to\infty.\end{aligned}$$

Thus, $\Theta_{\varpi}(\rho,\theta,\upsilon) = 1$. Similarly, We conclude that $\Theta_{\varpi}(\theta,\rho,\upsilon) = 1$ for $\upsilon \gg \vartheta$. This conclude that $\theta = \rho \in D\cap G$.

Now, we shall prove that (ρ,θ) is a strong CFP of Ξ. According to the Θ_{ϖ} triangularly property, for $\upsilon \gg \vartheta$.

By (*1*), (5) and (6), one can obtain

$$\begin{aligned}&\left(\frac{1}{\Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon)}-1\right)\\ &\le \left(\frac{1}{\Theta_{\varpi}(\rho,\rho_{i+1},\upsilon)}-1\right)\\ &\quad+\left(\frac{1}{\Theta_{\varpi}(\rho_{i+1},\Xi(\rho,\theta),\upsilon)}-1\right)\\ &= \left(\frac{1}{\Theta_{\varpi}(\rho,\rho_{i+1},\upsilon)}-1\right)\\ &\quad+\left(\frac{1}{\Theta_{\varpi}(\Xi(\theta_i,\rho_i),\Xi(\rho,\theta),\upsilon)}-1\right),\\ &\le \left(\frac{1}{\Theta_{\varpi}(\rho,\rho_{i+1},\upsilon)}-1\right)\\ &\quad+\gamma\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\theta_i,\Xi(\theta_i,\rho_i),\upsilon)}-1\\ +\frac{1}{\Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon)}-1\end{pmatrix}\\ &\quad+\delta\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\rho,\Xi(\theta_i,\rho_i),\upsilon)}-1\\ +\frac{1}{\Theta_{\varpi}(\theta_i,\Xi(\rho,\theta),\upsilon)}-1\end{pmatrix},\\ &\le (1+\delta)\left(\frac{1}{\Theta_{\varpi}(\rho,\rho_{i+1},\upsilon)}-1\right)\\ &\quad+\gamma\left(\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1\\ +\frac{1}{\Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon)}-1\end{pmatrix}\right)\\ &\quad+\delta\left(\frac{1}{\Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon)}-1\right)\\ &\to (\gamma+\delta)\left(\frac{1}{\Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon)}-1\right),\end{aligned} \tag{7}$$

as $i\to\infty$. Hence, (**??**) leads to $\Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon) = 1$, since $(\gamma+\delta) < \frac{1}{2}$, then $\Xi(\rho,\theta) = \rho = \theta$. Therefore (ρ,θ) is a strong CFP of Ξ.

The proof of the following corollaries follows immediately from Theorem *2*.

Corollary 1.*Suppose that D and G are two non-empty closed subsets of a CFCMS $(\Omega,\Theta_{\varpi},*)$, where Θ_{ϖ} is triangular and the mapping $\Xi : \Omega^3 \to \Omega$ is a cyclic coupled Kannan-type FCC w.r.t. D and G. Let Ξ verifies*

$$\begin{aligned}&\left(\frac{1}{\Theta_{\varpi}(\Xi(\theta,\rho),\Xi(q,s),\upsilon)}-1\right)\\ &\le \gamma\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\theta,\Xi(\theta,\rho),\upsilon)}-1\\ +\frac{1}{\Theta_{\varpi}(q,\Xi(q,s),\upsilon)}-1\end{pmatrix},\end{aligned}$$

where $\theta,s\in D$ and $\rho,q\in G$, for $\upsilon \gg \vartheta$,and $\gamma\in(0,\frac{1}{2})$. Then $D\cap G=\emptyset$ and Ξ has a strong CFP in $D\cap G$.

Corollary 2.*Suppose that D and G are two non-empty closed subsets of a CFCMS $(\Omega,\Theta_{\varpi},*)$, where Θ_{ϖ} is triangular and the mapping $\Xi : \Omega^3 \to \Omega$ is a cyclic coupled Chatterjea-type FCC w.r.t. D and G. Let Ξ verifies*

$$\begin{aligned}&\left(\frac{1}{\Theta_{\varpi}(\Xi(\theta,\rho),\Xi(q,s),\upsilon)}-1\right)\\ &\le \delta\begin{pmatrix}\frac{1}{\Theta_{\varpi}(\theta,\Xi(q,s),\upsilon)}-1\\ +\frac{1}{\Theta_{\varpi}(q,\Xi(\theta,\rho),\upsilon)}-1\end{pmatrix},\end{aligned}$$

where $\theta,s\in D$ and $\rho,q\in G$, for $\upsilon \gg \vartheta$,and $\delta\in(0,\frac{1}{2})$. Then $D\cap G=\emptyset$ and Ξ has a strong CFP in $D\cap G$.

To support Theorem *2*, we present the following example:

*Example 1.*Suppose that $\Omega=\mathbb{R}$ is a continuous $\upsilon-$norm and $\Xi:\Omega^2\to\Omega$ is described by

$$\Theta_{\varpi}(\theta,\rho,\upsilon) = \frac{\upsilon}{\upsilon+\varpi(\theta,\rho)},$$

where $\varpi(\theta,\rho) = |\theta-\rho|$ is a usual metric, for all $\theta,\rho\in\Omega$ and $\upsilon > \vartheta$. Then easily one can proved that $(\Omega,\Theta_{\varpi},*)$ is a CFCMS. Suppose that $D=[-1,0]$ and $G=[0,1]$ are two non-empty closed subsets of Ω with $\varpi(D,G)=0$. Define a continuous mapping $\Xi:\Omega^2\to\Omega$ by $\Xi(\theta,\rho)=\frac{-4\theta}{7}$. Then, the mapping Ξ is a cyclic mapping w.r.t. D and G for all $\theta,s\in D$ and $\rho,q\in G$. A mapping Ω is not a cyclic coupled Kannan-type contraction, since

$$\begin{aligned}\left(\frac{1}{\Theta_{\varpi}(\Xi(\theta,\rho),\Xi(q,s),\upsilon)}-1\right) &= \frac{1}{\upsilon}\varpi(\Xi(\theta,\rho),\Xi(q,s))\\ &= \frac{1}{\upsilon}\frac{4|\theta-q|}{7},\end{aligned}$$

where $\sigma=\frac{4}{7}\notin(0,\frac{1}{2})$, therefore Theorem *1 is not satisfied.*

Now, for $\upsilon \gg \vartheta$, we get

$$
\begin{aligned}
&\left(\frac{1}{\Theta_{\varpi}(\Xi(\theta,\rho),\Xi(q,s),\upsilon)}-1\right)\\
&=\frac{1}{\upsilon}\varpi(\Xi(\theta,\rho),\Xi(q,s))\\
&=\frac{1}{\upsilon}\frac{4|\theta-q|}{7}\le\frac{1}{\upsilon}\frac{4|\theta+q|}{7}\le\frac{1}{\upsilon}\frac{5|\theta+q|}{7}\\
&=\frac{1}{\upsilon}\left(\frac{4|\theta+q|}{7}+\frac{|\theta+q|}{7}\right)\\
&=\frac{4}{11\upsilon}\left|\frac{11\theta+11q}{7}\right|+\frac{1}{11\upsilon}\left|\frac{11\theta+11q}{7}\right|\\
&=\frac{1}{\upsilon}\left(\begin{array}{c}\frac{4}{11}\left(\left|\theta+\frac{4\theta}{7}+q+\frac{4q}{7}\right|\right)\\+\frac{1}{11}\left(\left|\theta+\frac{4q}{7}+q+\frac{4\theta}{7}\right|\right)\end{array}\right)\\
&\le\frac{1}{\upsilon}\left(\begin{array}{c}\frac{4}{11}\left(\left|\theta+\frac{4\theta}{7}\right|+\left|q+\frac{4q}{7}\right|\right)\\+\frac{1}{11}\left(\left|\theta+\frac{4q}{7}\right|+\left|q+\frac{4\theta}{7}\right|\right)\end{array}\right)\\
&=\frac{4}{11}\left(\begin{array}{c}\frac{1}{\Theta_{\varpi}(\theta,\Xi(\theta,\rho),\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(q,\Xi(q,s),\upsilon)}-1\end{array}\right)\\
&\quad+\frac{1}{11}\left(\begin{array}{c}\frac{1}{\Theta_{\varpi}(\theta,\Xi(q,s),\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(q,\Xi(\theta,\rho),\upsilon)}-1\end{array}\right).
\end{aligned}
$$

Hence, all requirements of Theorem 2 are satisfied with $\gamma=\frac{4}{11}$ and $\delta=\frac{1}{11}$ for $\upsilon \gg \vartheta$. Thus Ξ has a strong CFP, i.e., $\Xi(0,0)=0\in\mathbb{R}$.

The second result of this part is as follows:

Theorem 3. *Suppose that D and G are two non-empty closed subsets of a CFCMS $(\Omega,\Theta_{\varpi},*)$, where Θ_{ϖ} is triangular and the mapping $\Xi:\Omega^2\to\Omega$ is a cyclic coupled contractive-type mapping w.r.t. D and G verifying*

$$
\begin{aligned}
&\frac{1}{\Theta_{\varpi}(\Xi(\theta,\rho),\Xi(q,s),\upsilon)}-1\\
&\le\sigma\left(\frac{1}{\min\left\{\begin{array}{c}\Theta_{\varpi}(\theta,\Xi(\theta,\rho),\upsilon),\\ \Theta_{\varpi}(q,\Xi(q,s),\upsilon),\\ \Theta_{\varpi}(q,\Xi(\theta,\rho),\upsilon),\\ \Theta_{\varpi}(\theta,\Xi(q,s),\upsilon\end{array}\right\}}-1\right), \qquad (8)
\end{aligned}
$$

where $\theta,s\in D$ and $\rho,q\in G$, for $\upsilon\gg\vartheta$, and $\sigma\in[0,1)$. Then $D\cap G=\emptyset$ and Ξ has a strong CFP in $D\cap G$.

Proof. Define $\theta_0\in D$ and $\rho_0\in G$ Assume that $(\theta_i)\in G$ and $(\rho_i)\subset G$ are two sequences defined by (2), since Ω is a cyclic mapping w.r.t. D and G.

Now, we shall prove that (ρ_i) is a Cauchy sequence. For $i\ge\vartheta$,

$$
\begin{aligned}
&\left(\frac{1}{\Theta_{\varpi}(\rho_{k+1},\theta_{k+2},\upsilon)}-1\right)+\left(\frac{1}{\Theta_{\varpi}(\theta_{k+1},\rho_{k+2},\upsilon)}-1\right)\\
&\le\xi^{k+1}\left(\begin{array}{c}\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\end{array}\right),
\end{aligned}
$$

where $\xi=\frac{\sigma}{1-\sigma}<1$. First, we shall prove that

$$
\begin{aligned}
&\left(\frac{1}{\Theta_{\varpi}(\rho_{i+1},\theta_{i+2},\upsilon)}-1\right)\\
&\le\xi\left(\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1\right), \qquad (9)
\end{aligned}
$$

where $\xi=\frac{\sigma}{1-\sigma}<1$. Then, from (8), one can write

$$
\begin{aligned}
&\left(\frac{1}{\Theta_{\varpi}(\rho_{i+1},\theta_{i+2},\upsilon)}-1\right)\\
&=\left(\frac{1}{\Theta_{\varpi}(\Xi(\theta_i,\rho_i),\Xi(\rho_{i+1},\theta_{i+1}),\upsilon)}-1\right)\\
&\le\sigma\left(\frac{1}{\min\left\{\begin{array}{c}\Theta_{\varpi}(\theta_i,\Xi(\theta_i,\rho_i),\upsilon),\\ \Theta_{\varpi}(\rho_{i+1},\Xi(\rho_{i+1},\theta_{i+1}),\upsilon),\\ \Theta_{\varpi}(\theta_i,\Xi(\rho_{i+1},\theta_{i+1}),\upsilon),\\ \Theta_{\varpi}(\rho_{i+1},\Xi(\theta_i,\rho_i),\upsilon)\end{array}\right\}}-1\right)\\
&=\sigma\left(\frac{1}{\min\left\{\begin{array}{c}\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon),\\ \Theta_{\varpi}(\rho_{i+1},\theta_{i+2},\upsilon),\\ \Theta_{\varpi}(\theta_i,\theta_{i+2},\upsilon)\end{array}\right\}}-1\right). \qquad (10)
\end{aligned}
$$

Now, we have three cases:

(i) If $\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)$ is minimum, then $\left(\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1\right)$ is the maximum in (10), we have

$$
\begin{aligned}
&\left(\frac{1}{\Theta_{\varpi}(\rho_{i+1},\theta_{i+2},\upsilon)}-1\right)\\
&\le\sigma\left(\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1\right)\\
&\le\frac{\sigma}{1-\sigma}\left(\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1\right).
\end{aligned}
$$

It satisfies (9), as $\sigma<\frac{\sigma}{1-\sigma}$, where $\sigma\in[0,1)$.

(ii) If $\Theta_{\varpi}(\rho_{i+1},\theta_{i+2},\upsilon)$ is minimum, then $\left(\frac{1}{\Theta_{\varpi}(\rho_{i+1},\theta_{i+2},\upsilon)}-1\right)$ is the maximum in (10), so, we can write

$$
\left(\frac{1}{\Theta_{\varpi}(\rho_{i+1},\theta_{i+2},\upsilon)}-1\right)\le\sigma\left(\frac{1}{\Theta_{\varpi}(\rho_{i+1},\theta_{i+2},\upsilon)}-1\right),
$$

which is impossible.

(iii) If $\Theta_\varpi(\theta_i,\theta_{i+2},\upsilon)$ is minimum, then $\left(\frac{1}{\Theta_\varpi(\theta_i,\theta_{i+2},\upsilon)}-1\right)$ is the maximum in (10) so that

$$\begin{aligned}&\left(\frac{1}{\Theta_\varpi(\rho_{i+1},\theta_{i+2},\upsilon)}-1\right)\\ &\le\left(\frac{1}{\Theta_\varpi(\theta_i,\theta_{i+2},\upsilon)}-1\right)\\ &\le\left(\frac{1}{\Theta_\varpi(\theta_i,\rho_{i+1},\upsilon)}-1\right)+\left(\frac{1}{\Theta_\varpi(\rho_{i+1},\theta_{i+2},\upsilon)}-1\right)\\ &\le\frac{\sigma}{1-\sigma}\left(\frac{1}{\Theta_\varpi(\theta_i,\rho_{i+1},\upsilon)}-1\right).\end{aligned}$$

It follows that (9) fulfilled. Thus from all cases, we get that (9) is fulfilled.

Similarly, we can prove that

$$\begin{aligned}&\left(\frac{1}{\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)}-1\right)\\ &\le\xi\left(\frac{1}{\Theta_\varpi(\rho_i,\theta_{i+1},\upsilon)}-1\right),\end{aligned}\tag{11}$$

where $\xi=\frac{\sigma}{1-\sigma}<1$. Then, from (8), one can write

$$\begin{aligned}&\left(\frac{1}{\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)}-1\right)\\ &=\left(\frac{1}{\Theta_\varpi(\Xi(\rho_i,\theta_i),\Xi(\theta_{i+1},\rho_{i+1}),\upsilon)}-1\right)\\ &\le\sigma\left(\frac{1}{\min\left\{\begin{array}{c}\Theta_\varpi(\rho_i,\Xi(\rho_i,\theta_i),\upsilon),\\ \Theta_\varpi(\theta_{i+1},\Xi(\theta_{i+1},\rho_{i+1}),\upsilon),\\ \Theta_\varpi(\rho_i,\Xi(\theta_{i+1},\rho_{i+1}),\upsilon),\\ \Theta_\varpi(\theta_{i+1},\Xi(\rho_i,\theta_i),\upsilon)\end{array}\right\}}-1\right)\\ &=\sigma\left(\frac{1}{\min\left\{\begin{array}{c}\Theta_\varpi(\rho_i,\theta_{i+1},\upsilon),\\ \Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon),\\ \Theta_\varpi(\rho_i,\rho_{i+2},\upsilon)\end{array}\right\}}-1\right).\end{aligned}\tag{12}$$

Hence, again we have three cases:

(i) If $\Theta_\varpi(\rho_i,\theta_{i+1},\upsilon)$ is minimum, then $\left(\frac{1}{\Theta_\varpi(\rho_i,\theta_{i+1},\upsilon)}-1\right)$ is the maximum in (12), we get

$$\begin{aligned}&\left(\frac{1}{\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)}-1\right)\\ &\le\sigma\left(\frac{1}{\Theta_\varpi(\rho_i,\theta_{i+1},\upsilon)}-1\right)\\ &\le\frac{\sigma}{1-\sigma}\left(\frac{1}{\Theta_\varpi(\rho_i,\theta_{i+1},\upsilon)}-1\right).\end{aligned}$$

It satisfies (11), as $\sigma<\frac{\sigma}{1-\sigma}$, where $\sigma\in[0,1)$.

(ii) If $\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)$ is minimum, then $\left(\frac{1}{\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)}-1\right)$ is the maximum in (12), we get

$$\begin{aligned}&\left(\frac{1}{\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)}-1\right)\\ &\le\sigma\left(\frac{1}{\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)}-1\right),\end{aligned}$$

which is impossible.

(iii) If $\Theta_\varpi(\rho_i,\rho_{i+2},\upsilon)$ is minimum, then $\left(\frac{1}{\Theta_\varpi(\rho_i,\rho_{i+2},\upsilon)}-1\right)$ is the maximum in (12), we get

$$\begin{aligned}&\left(\frac{1}{\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)}-1\right)\\ &\le\left(\frac{1}{\Theta_\varpi(\rho_i,\rho_{i+2},\upsilon)}-1\right)\\ &\le\sigma\left(\begin{array}{c}\frac{1}{\Theta_\varpi(\rho_i,\theta_{i+1},\upsilon)}-1\\ +\frac{1}{\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)}-1\end{array}\right)\\ &\le\frac{\sigma}{1-\sigma}\left(\frac{1}{\Theta_\varpi(\rho_i,\theta_{i+1},\upsilon)}-1\right).\end{aligned}$$

It follows that (11) justified. Thus, from all cases, we get that (11) is fulfilled.

Now, by adding (9) and (11), we can write

$$\begin{aligned}&\left(\frac{1}{\Theta_\varpi(\rho_{i+1},\theta_{i+2},\upsilon)}-1\right)+\left(\frac{1}{\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)}-1\right)\\ &\le\xi\left(\begin{array}{c}\left(\frac{1}{\Theta_\varpi(\theta_i,\rho_{i+1},\upsilon)}-1\right)\\ +\left(\frac{1}{\Theta_\varpi(\rho_i,\theta_{i+1},\upsilon)}-1\right)\end{array}\right).\end{aligned}\tag{13}$$

Now, again by (8) and similar as above, we obtain

$$\begin{aligned}&\left(\frac{1}{\Theta_\varpi(\theta_i,\rho_{i+1},\upsilon)}-1\right)\\ &\le\xi\left(\frac{1}{\Theta_\varpi(\rho_{i-1},\theta_i,\upsilon)}-1\right),\end{aligned}\tag{14}$$

$$\begin{aligned}&\left(\frac{1}{\Theta_\varpi(\rho_i,\theta_{i+1},\upsilon)}-1\right)\\ &\le\xi\left(\frac{1}{\Theta_\varpi(\theta_{i-1},\rho_i,\upsilon)}-1\right),\end{aligned}\tag{15}$$

where $\xi=\frac{\sigma}{1-\sigma}<1$. Hence, again by adding (14) and (15) and then putting in (13), we get

$$\begin{aligned}&\left(\frac{1}{\Theta_\varpi(\rho_{i+1},\theta_{i+2},\upsilon)}-1\right)+\left(\frac{1}{\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)}-1\right)\\ &\le\xi^2\left(\begin{array}{c}\left(\frac{1}{\Theta_\varpi(\rho_{i-1},\theta_i,\upsilon)}-1\right)\\ +\left(\frac{1}{\Theta_\varpi(\theta_{i-1},\rho_i,\upsilon)}-1\right)\end{array}\right),\end{aligned}$$

by continuing, we get

$$\left(\frac{1}{\Theta_\varpi(\rho_{i+1},\theta_{i+2},\upsilon)}-1\right)+\left(\frac{1}{\Theta_\varpi(\theta_{i+1},\rho_{i+2},\upsilon)}-1\right)$$
$$\leq \xi^{i+1}\left(\begin{array}{c}\frac{1}{\Theta_\varpi(\rho_0,\theta_1,\upsilon)}-1\\+\frac{1}{\Theta_\varpi(\theta_0,\rho_1,\upsilon)}-1\end{array}\right), \tag{16}$$

for $i \geq \vartheta$. Now, for integer k, we get

$$\left(\frac{1}{\Theta_\varpi(\rho_{k+1},\theta_{k+1},\upsilon)}-1\right)$$
$$=\left(\frac{1}{\Theta_\varpi(\Xi(\theta_k,\rho_k),\Xi(\rho_k,\theta_k),\upsilon)}-1\right)$$
$$\leq \sigma\left(\frac{1}{\min\left\{\begin{array}{c}\Theta_\varpi(\theta_k,\Xi(\theta_k,\rho_k),\upsilon),\\\Theta_\varpi(\rho_k,\Xi(\rho_k,\theta_k),\upsilon),\\\Theta_\varpi(\theta_k,\Xi(\rho_k,\theta_k),\upsilon),\\\Theta_\varpi(\rho_k,\Xi(\theta_k,\rho_k),\upsilon)\end{array}\right\}}-1\right)$$
$$=\sigma\left(\frac{1}{\min\left\{\begin{array}{c}\Theta_\varpi(\theta_k,\rho_{k+1},\upsilon),\\\Theta_\varpi(\rho_k,\theta_{k+1},\upsilon),\\\Theta_\varpi(\theta_k,\theta_{k+1},\upsilon),\\\Theta_\varpi(\rho_k,\rho_{k+1},\upsilon)\end{array}\right\}}-1\right). \tag{17}$$

Hence, we the cases below:

(a) If $\Theta_\varpi(\theta_k,\rho_{k+1},\upsilon)$ is minimum, then $\left(\frac{1}{\Theta_\varpi(\theta_k,\rho_{k+1},\upsilon)}-1\right)$ is the maximum in (17), we get

$$\left(\frac{1}{\Theta_\varpi(\rho_{k+1},\theta_{k+1},\upsilon)}-1\right)$$
$$\leq \sigma\left(\frac{1}{\Theta_\varpi(\theta_k,\rho_{k+1},\upsilon)}-1\right)$$
$$\leq \xi\left(\frac{1}{\Theta_\varpi(\theta_k,\rho_{k+1},\upsilon)}-1\right),$$

where $\sigma<\xi=\frac{\sigma}{1-\sigma}<1$, since $\sigma\in[0,1)$.

(b) If $\Theta_\varpi(\rho_k,\theta_{k+1},\upsilon)$ is minimum, then $\left(\frac{1}{\Theta_\varpi(\rho_k,\theta_{k+1},\upsilon)}-1\right)$ is the maximum in (17), we get

$$\left(\frac{1}{\Theta_\varpi(\rho_{k+1},\theta_{k+1},\upsilon)}-1\right)$$
$$\leq \sigma\left(\frac{1}{\Theta_\varpi(\rho_k,\theta_{k+1},\upsilon)}-1\right)$$
$$\leq \xi\left(\frac{1}{\Theta_\varpi(\rho_k,\theta_{k+1},\upsilon)}-1\right),$$

where $\sigma<\xi=\frac{\sigma}{1-\sigma}<1$, since $\sigma\in[0,1)$.

(c) If $\Theta_\varpi(\rho_k,\rho_{k+1},\upsilon)$ is minimum, then $\left(\frac{1}{\Theta_\varpi(\rho_k,\rho_{k+1},\upsilon)}-1\right)$ is the maximum in (17), we get

$$\left(\frac{1}{\Theta_\varpi(\rho_{k+1},\theta_{k+1},\upsilon)}-1\right)$$
$$\leq \sigma\left(\frac{1}{\Theta_\varpi(\rho_k,\rho_{k+1},\upsilon)}-1\right)$$
$$\leq \sigma\left(\begin{array}{c}\frac{1}{\Theta_\varpi(\rho_k,\theta_{k+1},\upsilon)}-1\\+\frac{1}{\Theta_\varpi(\theta_{k+1},\rho_{k+1},\upsilon)}-1\end{array}\right)$$
$$\leq \xi\left(\frac{1}{\Theta_\varpi(\rho_k,\theta_{k+1},\upsilon)}-1\right),$$

where $\sigma<\xi=\frac{\sigma}{1-\sigma}<1$, since $\sigma\in[0,1)$.

(d) If $\Theta_\varpi(\theta_k,\theta_{k+1},\upsilon)$ is minimum, then $\left(\frac{1}{\Theta_\varpi(\theta_k,\theta_{k+1},\upsilon)}-1\right)$ is the maximum in (17), we get

$$\left(\frac{1}{\Theta_\varpi(\rho_{k+1},\theta_{k+1},\upsilon)}-1\right)$$
$$\leq \sigma\left(\begin{array}{c}\frac{1}{\Theta_\varpi(\theta_k,\rho_{k+1},\upsilon)}-1\\+\frac{1}{\Theta_\varpi(\rho_{k+1},\theta_{k+1},\upsilon)}-1\end{array}\right)$$
$$\leq \xi\left(\frac{1}{\Theta_\varpi(\theta_k,\rho_{k+1},\upsilon)}-1\right),$$

where $\sigma<\xi=\frac{\sigma}{1-\sigma}<1$, since $\sigma\in[0,1)$.

Then, from (a) and (d), we can write

$$\left(\frac{1}{\Theta_\varpi(\rho_{k+1},\theta_{k+1},\upsilon)}-1\right)$$
$$\leq \xi\left(\frac{1}{\Theta_\varpi(\theta_k,\rho_{k+1},\upsilon)}-1\right), \tag{18}$$

where $\xi=\frac{\sigma}{1-\sigma}<1$. And from (b) and (c), we can write

$$\left(\frac{1}{\Theta_\varpi(\rho_{k+1},\theta_{k+1},\upsilon)}-1\right)$$
$$\leq \xi\left(\frac{1}{\Theta_\varpi(\rho_k,\theta_{k+1},\upsilon)}-1\right), \tag{19}$$

where $\xi=\frac{\sigma}{1-\sigma}<1$. Now, by adding (18) and (19), we can write

$$\left(\frac{1}{\Theta_\varpi(\rho_{k+1},\theta_{k+1},\upsilon)}-1\right)$$
$$\leq \psi\left(\frac{1}{\Theta_\varpi(\rho_k,\theta_{k+1},\upsilon)}-1+\frac{1}{\Theta_\varpi(\theta_k,\rho_{k+1},\upsilon)}-1\right),$$

where $\psi=\frac{\xi}{2}$. Hence, in view of (16), one can get

$$\left(\frac{1}{\Theta_\varpi(\rho_{k+1},\theta_{k+1},\upsilon)}-1\right)$$
$$\leq \psi\xi^k\left(\begin{array}{c}\frac{1}{\Theta_\varpi(\rho_0,\theta_1,\upsilon)}-1\\+\frac{1}{\Theta_\varpi(\theta_0,\rho_1,\upsilon)}-1\end{array}\right), \tag{20}$$

for $k \geq 0$. By triangular inequality (16) and (20) for $i \geq \vartheta$, we get

$$\left(\frac{1}{\Theta_{\varpi}(\rho_i,\rho_{i+1},\upsilon)}-1+\frac{1}{\Theta_{\varpi}(\theta_i,\theta_{i+1},\upsilon)}-1\right)$$
$$\leq \left(\frac{1}{\Theta_{\varpi}(\rho_i,\theta_i,\upsilon)}-1+\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1\right)$$
$$+\left(\frac{1}{\Theta_{\varpi}(\theta_i,\rho_i,\upsilon)}-1+\frac{1}{\Theta_{\varpi}(\rho_i,\theta_{i+1},\upsilon)}-1\right)$$

$$=\left(\frac{1}{\Theta_{\varpi}(\rho_i,\theta_i,\upsilon)}-1+\frac{1}{\Theta_{\varpi}(\theta_i,\rho_i,\upsilon)}-1\right)$$
$$+\left(\frac{1}{\Theta_{\varpi}(\theta_i,\rho_{i+1},\upsilon)}-1+\frac{1}{\Theta_{\varpi}(\rho_i,\theta_{i+1},\upsilon)}-1\right)$$

$$\leq 2\psi\xi^{i-1}\left(\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1+\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\right)$$
$$+\xi^{i}\left(\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1+\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\right)$$
$$=\left(1+\frac{2\psi}{\xi}\right)\xi^{i}\left(\begin{array}{c}\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\end{array}\right). \tag{21}$$

Now, for $i,j \geq \vartheta$ and $j>i$, we get

$$\left(\frac{1}{\Theta_{\varpi}(\rho_i,\rho_j,\upsilon)}-1\right)$$
$$\leq \sum_{k=i}^{j-1}\left(\frac{1}{\Theta_{\varpi}(\rho_k,\rho_{k+1},\upsilon)}-1\right)$$
$$\leq \sum_{k=i}^{j-1}\left(1+\frac{2\psi}{\xi}\right)\xi^{k}\left(\begin{array}{c}\left(\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1\right)\\+\left(\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\right)\end{array}\right)$$

$$=\left(1+\frac{2\psi}{\xi}\right)\frac{\xi^{k}}{1-\xi}\left(\begin{array}{c}\left(\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1\right)\\+\left(\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\right)\end{array}\right)$$
$$\to \vartheta, \text{ as } i\to\infty.$$

Hence, we proved that (ρ_i) is a Cauchy sequence and it convergent in Ω.

Where D and G are a closed subsets of Ω, so that

$$\rho_i \to \rho \in G, \text{ as } i\to\infty. \tag{22}$$

Analogously,

$$\theta_i \to \theta \in D, \text{ as } i\to\infty. \tag{23}$$

Therefore, from (22) and (23), we get

$$\lim_{i\to\infty}\Theta_{\varpi}(\rho_i,\theta_i,\upsilon)=\Theta_{\varpi}(\rho,\theta,\upsilon), \text{and}$$
$$\lim_{i\to\infty}\Theta_{\varpi}(\theta_i,\rho_i,\upsilon)=\Theta_{\varpi}(\theta,\rho,\upsilon).$$

By triangular inequality (16) and (21), we get

$$\left(\frac{1}{\Theta_{\varpi}(\rho_i,\theta_i,\upsilon)}-1\right)$$
$$\leq \left(\frac{1}{\Theta_{\varpi}(\rho_i,\rho_{i+1},\upsilon)}-1\right)+\left(\frac{1}{\Theta_{\varpi}(\rho_{i+1},\theta_i,\upsilon)}-1\right)$$
$$\leq \left(\frac{\xi+2\psi}{\xi}+1\right)\xi^{k}\left(\begin{array}{c}\frac{1}{\Theta_{\varpi}(\rho_0,\theta_1,\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\theta_0,\rho_1,\upsilon)}-1\end{array}\right)$$
$$\to \vartheta, \text{ as } i\to\infty.$$

Hence, $\Theta_{\varpi}(\rho,\theta,\upsilon)=1$, which leads to $\Xi(\rho,\theta)=\rho=\theta\in D\cap G$.

Now, we can prove that Ξ has a strong CFP in $D\cap G$,

$$\left(\frac{1}{\Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon)}-1\right)$$
$$\leq \left(\begin{array}{c}\frac{1}{\Theta_{\varpi}(\rho,\rho_{i+1},\upsilon)}-1\\+\frac{1}{\Theta_{\varpi}(\rho_{i+1},\Xi(\rho,\theta),\upsilon)}-1\end{array}\right) \tag{24}$$

Therefore, by the view of (8), (22) and (23), one can write

$$\frac{1}{\Theta_{\varpi}(\Xi(\theta_i,\rho_i),\Xi(\rho,\theta),\upsilon)}-1$$
$$\leq \sigma\left(\frac{1}{\min\left\{\begin{array}{c}\Theta_{\varpi}(\theta_i,\Xi(\theta_i,\rho_i),\upsilon),\\ \Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon),\\ \Theta_{\varpi}(\rho,\Xi(\theta_i,\rho_i),\upsilon),\\ \Theta_{\varpi}(\theta_i,\Xi(\rho,\theta),\upsilon)\end{array}\right\}}-1\right)$$

$$=\sigma\left(\frac{1}{\min\left\{\begin{array}{c}\Theta_{\varpi}(\theta_i,\rho_{i+1}),\upsilon),\\ \Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon),\\ \Theta_{\varpi}(\rho,\rho_{i+1}),\upsilon),\\ \Theta_{\varpi}(\theta_i,\Xi(\rho,\theta),\upsilon)\end{array}\right\}}-1\right)$$
$$\to \sigma\left(\frac{1}{\Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon)}-1\right), \text{ as } i\to\infty. \tag{25}$$

Hence, from (24), we have

$$\left(\frac{1}{\Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon)}-1\right)$$
$$\leq \left(\frac{1}{\Theta_{\varpi}(\rho,\rho_{i+1},\upsilon)}-1+\frac{1}{\Theta_{\varpi}(\rho_{i+1},\Xi(\rho,\theta),\upsilon)}-1\right)$$
$$\to \sigma\left(\frac{1}{\Theta_{\varpi}(\rho,\Xi(\rho,\theta),\upsilon)}-1\right), \text{ as } i\to\infty,$$

which verifies that $\Theta_{\varpi}(\rho,\theta,\upsilon)=1$, where $1-\sigma\neq\vartheta$. Thus, $\Xi(\rho,\theta)=\rho=\theta$, which implies that $\Xi(\rho,\theta)$ is a strong CFP of Ξ.

Corollary 3. *Suppose that D and G are two non-empty closed subsets of a CFCMS $(\Omega,\Theta_{\varpi},*)$, where Θ_{ϖ} is traingular and the mapping $\Xi:\Omega^2\to\Omega$ is a cyclic coupled contractive-type mapping w.r.t. D and G verifying*

$$\frac{1}{\Theta_{\varpi}(\Xi(\theta,\rho),\Xi(q,s),\upsilon)}-1 \leq \sigma\left(\frac{1}{\min\left\{\begin{array}{c}\Theta_{\varpi}(\theta,\Xi(\theta,\rho),\upsilon),\\ \Theta_{\varpi}(q,\Xi(q,s),\upsilon)\end{array}\right\}}-1\right),$$

where $\theta,s\in D$ and $\rho,q\in G$, for $\upsilon\gg\vartheta$, and $\sigma\in[0,1)$. Then $D\cap G=\emptyset$ and Ξ has a strong CFP in $D\cap G$.

The following example support Theorem 3:

Example 2. Assume that all requirements of Example 1 hold. Define the mapping $\Xi:\Omega^2\to\Omega$ by $\Xi(\theta,\rho)=\frac{-2\theta}{5}$. Then, the mapping Ξ is a cyclic mapping w.r.t. D and G for all $\theta,s\in D$ and $\rho,q\in G$. Now, from for $\upsilon\gg\vartheta$, we have

$$\begin{aligned}
&\frac{1}{\Theta_{\varpi}(\Xi(\theta,\rho),\Xi(q,s),\upsilon)}-1\\
&=\frac{1}{\upsilon}\varpi(\Xi(\theta,\rho),\Xi(q,s))\\
&=\frac{1}{\upsilon}\frac{2|\theta-q|}{5}\leq\frac{1}{\upsilon}\frac{14|\theta-q|}{25}\\
&\leq\frac{1}{\upsilon}\cdot\frac{2}{5}\cdot\frac{7}{5}\left(\max\left\{\theta,q,\frac{5\theta+2q}{7},\frac{5q+\theta}{7}\right\}\right)\\
&=\frac{2}{5\upsilon}\left(\max\left\{\frac{7\theta}{5},\frac{7q}{5},\frac{5\theta+2q}{5},\frac{5q+\theta}{5}\right\}\right)\\
&\leq\frac{2}{5\upsilon}\left(\max\left\{\begin{array}{c}\left|\theta+\frac{2\theta}{5}\right|,\left|q+\frac{2q}{5}\right|,\\ \left|\theta+\frac{2q}{5}\right|,\left|q+\frac{2\theta}{5}\right|\end{array}\right\}\right)\\
&=\frac{2}{5}\left(\frac{1}{\min\left\{\begin{array}{c}\Theta_{\varpi}(\theta,\Xi(\theta,\rho),\upsilon),\\ \Theta_{\varpi}(q,\Xi(q,s),\upsilon),\\ \Theta_{\varpi}(q,\Xi(\theta,\rho),\upsilon),\\ \Theta_{\varpi}(\theta,\Xi(q,s),\upsilon\end{array}\right\}}-1\right).
\end{aligned}$$

Hence, all requirements of Theorem 3 are justified with $\sigma=\frac{2}{5}$ for $\upsilon\gg\vartheta$. Then Ξ has a strong CFP, i.e. $\Xi(0,0)=0\in\mathbb{R}$.

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